Long-time Behavior of Random Walks in Random Environment and Asymptotics of a Ternary Coalescent Process

Dissertation

zur

Erlangung der naturwissenschaftlichen Doktorwürde
(Dr. sc. nat.)

vorgelegt der

Mathematisch-naturwissenschaftlichen Fakultät

der

Universität Zürich

von

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Zürich, 2013

Summary

The first part of this thesis deals with random walks in random environment on the d-dimensional lattice $\mathbb{Z}^d$, $d \geq 3$. Such stochastic systems can be used to model the random motion of a particle in an inhomogeneous medium. The effects of irregularities caused by impurities or defects of the medium are from a mathematical point of view best described by randomizing the medium. In fact, randomness enters at two different levels: It governs the choice of the environment and also the movement of the particle.

In our case, the random environment is modeled by independent and identically distributed random transition kernels $(\omega_x(e))_{|e|=1,\, e\in\mathbb{Z}^d}$, $x \in \mathbb{Z}^d$, which are small isotropic perturbations of the homogeneous simple random walk kernel $p(x, x+\cdot) \equiv 1/(2d)$. Given an environment ω, one considers the random walk with transition kernel $p_\omega(x, x+\cdot) = \omega_x(\cdot)$. First, we investigate exit distributions of such walks from large balls. We show that when the radius of the ball tends to infinity, the exit measure is approximately given by that of a simple random walk. More precisely, we transfer estimates on the total variation distance between these two measures and smoothed versions thereof from smaller to larger radii. Further, we compare the exit distributions on certain boundary portions. Finally, under an additional assumption on the measure governing the environment, we use the information on the spatial behavior to control mean sojourn times in large balls.

In the second part of the thesis, from page 91 onwards, we consider coagulation and fragmentation processes. Fragmentation processes describe a memoryless evolution of particles characterized by their masses, which split independently into (smaller) new particles. Conversely, coagulation or coalescent processes model the coagulation of particles, where the rate at which a family merges depends only on the members involved. One again assumes that the particles are determined by their masses, and that the system develops in a Markovian way.

We study a ternary coalescent process where three particles of masses $r, s, t > 0$ coagulate at rate $r + s + t + 3$. Different representations in terms of quantities related to one-dimensional simple random walk and random binary forests are used to establish various properties of this process. First we show that time reversal results in a fragmentation process. Then we investigate asymptotic behavior. Starting from a fixed number of N particles of unit mass we let N tend to infinity and obtain under an appropriate rescaling a well-known binary coalescent process, the so-called standard additive coalescent. Finally, we look at particle densities and solve the associated Smoluchowski coagulation equations.

Zusammenfassung

Im ersten Teil der vorliegenden Dissertation beschäftigen wir uns mit Irrfahrten in einer zufälligen Umgebung auf dem d-dimensionalen Gitter $\mathbb{Z}^d$, $d \geq 3$. Damit kann die Diffusion eines Teilchen in einem inhomogenen Medium, zum Beispiel einem porösen Gestein, modelliert werden. Mathematisch können die Auswirkungen von Unregelmäßigkeiten im Material am besten stochastisch beschrieben werden. Nicht mehr nur das Teilchen selbst

führt eine zufällige Bewegung aus, sondern auch die Struktur der Umgebung wird durch eine Wahrscheinlichkeitsverteilung beschrieben.

Wir modellieren die zufällige Umgebung durch eine Familie von gleichverteilten unabhängigen Übergangskernen $(\omega_x(e))_{|e|=1,\, e\in\mathbb{Z}^d}$, $x \in \mathbb{Z}^d$. Von der zugrunde liegenden Verteilung nehmen wir an, dass sie invariant ist unter orthogonalen Abbildungen, die das Gitter $\mathbb{Z}^d$ auf sich selbst abbilden. Weiter fordern wir, dass sie konzentriert ist auf Übergangswahrscheinlichkeiten, die nur wenig von dem homogenen Übergangskern $p(x, x + \cdot) \equiv 1/(2d)$ der einfachen Irrfahrt abweichen. Für eine feste Umgebung ω betrachtet man nun die Irrfahrt mit Übergangswahrscheinlichkeiten $p_\omega(x, x + \cdot) = \omega_x(\cdot)$.

Zunächst untersuchen wir Austrittsmaße solcher Irrfahrten aus großen Kugeln. Wir vergleichen sie mit der entsprechenden Austrittsverteilung der einfachen Irrfahrt. Genauer zeigen wir, wie sich Abschätzungen in der Variationsnorm für Kugeln eines bestimmten Radius auf Kugeln größeren Radien übertragen lassen. Zudem erhalten wir für eine große Klasse von Umgebungen lokale Informationen über die Massenverteilung des Austrittsmaßes. Unter einer zusätzlichen Annahme an die Verteilung der Umgebung kontrollieren wir schließlich mittlere Aufenthaltszeiten in großen Kugeln.

Im zweiten Teil der Dissertation ab Seite 91 betrachten wir Fragmentations- und Koagulationsprozesse. Erstere beschreiben das zeitliche Verhalten eines Teilchensystems, in dem sich einzelne Teilchen unabhängig voneinander in kleinere Teilchen aufspalten können. Man nimmt zur Vereinfachung an, dass die Teilchen nur durch ihre Massen bestimmt sind, und dass nur der gegenwärtige Zustand Einfluss auf das zukünftige Verhalten hat (Markoffeigenschaft). Im Gegensatz dazu beschreiben Koagulationsprozesse Systeme, in denen sich Teilchen zu einem neuen Teilchen zusammenschließen können, wobei sich die Massen aufaddieren. Man nimmt wieder an, dass sich das System gedächtnislos verhält, die Teilchen durch ihre Masse bestimmt sind und dass die Koagulationsraten nur von den sich vereinigenden Teilchenmassen abhängen.

Wir behandeln einen speziellen ternären Koagulationsprozess, bei dem sich drei Teilchen der Massen $r, s, t > 0$ mit Rate $r + s + t + 3$ zu einem neuen Teilchen zusammenschließen. Wir zeigen, dass dieser Prozess verschiedene Darstellungen besitzt, die eine Untersuchung wichtiger Eigenschaften ermöglichen. Zum Beispiel erhält man durch Zeitumkehr einen Fragmentationsprozess. Weiter beweisen wir, dass ein System von N Teilchen der Masse 1 für $N \to \infty$ gegen einen bekannten binären Koagulationsprozess konvergiert, den sogenannten standard additive coalescent. Im letzten Teil betrachten wir Teilchendichten und lösen die zugehörigen Koagulationsgleichungen von Smoluchowski explizit.

Danksagung

Mein herzlicher Dank gilt meinen beiden Betreuern, Prof. Dr. Erwin Bolthausen und Prof. Dr. Jean Bertoin. Von zwei solch eminenten Mathematikern zu lernen ist Vergnügen, Chance und Verpflichtung zugleich.

Mit Erwin Bolthausen habe ich an Irrfahrten in zufälliger Umgebung gearbeitet. Dass wir am Ende nicht ganz unser urspüngliches Ziel erreicht haben, werte ich als ein gutes Zeichen: Es lädt ein zu einer weiteren gemeinsamen Beschäftigung über die Doktorarbeit hinaus, auf die ich mich sehr freue.

Im dritten Jahr meiner Dissertation habe ich die Möglichkeit bekommen, mit Jean Bertoin zusammenzuarbeiten. Für mich war dies eine einmalige Chance, von der ich bis heute sehr profitiere. Es macht mich glücklich zu wissen, dass unsere Zusammenarbeit noch nicht beendet ist.

Mein Dank geht auch an Prof. Dr. Ofer Zeitouni für spannende Diskussionen sowie für die Bereitschaft, diese Dissertation zu begutachten.

Weiter möchte ich mich bei meinen Freunden am Institut bedanken - Alessandra, Julia, Steffen, um nur drei zu nennen. Es haben sich einige Freundschaften ergeben, die über die gemeinsame Zeit in Zürich hinaus Bestand haben werden.

Schließlich gilt mein großer Dank meinen Eltern für ihre Unterstützung über all die Jahre, in so vielerlei Hinsicht.

Zürich, im März 2013 Erich Baur

Part 1: Random Walks in Random Environment

Contents of Part 1

0 Introduction

0.1 The model

General description

Consider the integer lattice $\mathbb{Z}^d$ with unit vectors e_i, whose ith component equals 1. We let $\mathcal{P}$ be the set of probability distributions on $\{\pm e_i : i = 1, \dots, d\}$. Given a probability measure μ on $\mathcal{P}$, we equip $\Omega = \mathcal{P}^{\mathbb{Z}^d}$ with its natural product σ-field $\mathcal{F}$ and the product measure $\mathbb{P}_\mu = \mu^{\otimes \mathbb{Z}^d}$. Each element $\omega \in \Omega$ yields transition probabilities of a nearest neighbor Markov chain on $\mathbb{Z}^d$, the *random walk in random environment* (RWRE for short), via

$$p_\omega(x, x+e) = \omega_x(e), \quad e \in \{\pm e_i : i = 1, \dots, d\}.$$

We write $\mathrm{P}_{x,\omega}$ for the "quenched" law of the canonical Markov chain $(X_n)_{n\geq 0}$ with these transition probabilities, starting at $x \in \mathbb{Z}^d$. The probability measure

$$P = \int_\Omega \mathrm{P}_{0,\omega}\, \mathbb{P}(\mathrm{d}\omega)$$

is commonly referred to as averaged or "annealed" law of the RWRE started at the origin.

Additional requirements

We study asymptotic properties of the RWRE in dimension $d \geq 3$ when the underlying environments are small perturbations of the fixed environment $\omega_x(\pm e_i) = 1/(2d)$ corresponding to simple or standard random walk. In order to fix a perturbative regime, we introduce the following condition.

- Let $0 < \varepsilon < 1/(2d)$. We say that $\mathbf{A0}(\varepsilon)$ holds if $\mu(\mathcal{P}_\varepsilon) = 1$, where

$$\mathcal{P}_\varepsilon = \{q \in \mathcal{P} : |q(\pm e_i) - 1/(2d)| \leq \varepsilon \text{ for all } i = 1, \dots, d\}.$$

The perturbative behavior concerns the behavior of the RWRE when $\mathbf{A0}(\varepsilon)$ holds for small ε. However, even for arbitrarily small ε, such walks can behave very differently compared to simple random walk. This motivates a further "centering" restriction on μ.

- We say that **A1** holds if μ is invariant under all orthogonal transformations fixing the lattice $\mathbb{Z}^d$, i.e. if $O : \mathbb{R}^d \to \mathbb{R}^d$ is any orthogonal matrix that maps $\mathbb{Z}^d$ onto itself, then the laws of $(\omega_0(Oe))_{|e|=1}$ and $(\omega_0(e))_{|e|=1}$ coincide.

If **A1** holds, $\mathbb{P}_\mu$ is called *isotropic.*

0.2 Informal description of the results

In the following, we write $\mathbb{P}$ instead of $\mathbb{P}_\mu$ and denote by $\mathbb{E}$ the corresponding expectation.

Exit laws from balls

In the main part of this work, we investigate the RWRE exit distribution from the ball $V_L = \{x \in \mathbb{Z}^d : |x| \leq L\}$ when the radius L is large. Assuming **A1** and **A0**(ε) for small ε, we show that the exit law of the walk, started from a point x with $|x| \leq L/5$, is close to that of simple random walk. More precisely, using the multiscale analysis introduced in Bolthausen and Zeitouni [6], we prove that if the radius L tends to infinity, then

(i) The difference of the two exit laws measured in total variation stays small as L increases (but does not tend to zero, due to boundary effects) (Theorem 1.1 (i)).

(ii) The distance between the two exit laws converges to zero if they are convolved with an additional smoothing kernel on a scale increasing arbitrarily slowly with L (Theorem 1.1 (ii)).

(iii) The RWRE exit measure of boundary portions of size $\geq (L/(\log L)^{15})^{d-1}$ can be bounded from above by that of simple random walk. Evaluated on segments of size $\geq (L/(\log L)^6)^{d-1}$, the two measures agree up to a multiplicative error, which tends to one as L increases (Theorem 1.2).

The first two parts already appeared in [6], which serves as the basis for our work. However, for reasons explained below, it was of great interest to find a somewhat different approach.

Mean sojourn times

The results on exit laws can be used to prove transience of the RWRE (Corollary 1.1), and they provide an invariance principle up to time transformation. Getting complete control over time is a major open problem, and in that direction, we look in Section 8 at mean holding or sojourn times in balls. Our basic insight is that exceptionally small or large times can only be produced by spatially atypical regions. Consequently, the philosophy behind our approach is to derive statements on sojourn times from estimates on exit laws. However, our results on exit distributions seem not quite sufficient to handle the presence of strong traps, i.e. regions where the RWRE cannot escape for a long time with high probability. We therefore make an additional assumption which guarantees that the mass of environments producing very large times is sufficiently small. Let $\tau_L = \inf\{n \geq 0 : X_n \notin V_L\}$ be the first exit time of the RWRE from the ball V_L, and denote by $\mathrm{E}_{0,\omega}$ the expectation with respect to $\mathrm{P}_{0,\omega}$.

- We say that **A2** holds if for large L,

$$\mathbb{P}\left(\mathrm{E}_{0,\omega}\left[\tau_L\right] > (\log L)^4 L^2\right) \leq L^{-8d}.$$

Assuming this additional condition, we prove

(iv) For almost all environments, the normalized quenched mean time $\mathrm{E}_{0,\omega}[\tau_L]/L^2$ is finally contained in a small interval around one, where the size of the interval converges to zero if the disorder ε tends to zero (Proposition 1.2 and Theorem 1.3).

We believe that **A2** follows from $\mathbf{A0}(\varepsilon)$ and **A1**, even with a faster decay of the probability. It remains an open (and possibly challenging) problem to prove this. An example where **A2** trivially holds true is given in Remark 8.1.

0.3 Discussion of this work

The part on exit measures should be seen as a corrected and extended version of Bolthausen and Zeitouni [6]. Most of the ideas can already be found there, and also our proofs sometimes follow those of [6]. However, our focus lies more on Green's function estimates on "goodified" environments, which are developed in Section 4. Partly based on (unpublished) notes of Bolthausen, this section is entirely new, and the results obtained make the proofs of the main statements more transparent. The core statement is Lemma 4.1, which gives a bound on the (coarse grained) RWRE Green's function, for a large class of environments. As such estimates were only partially present in [6], the authors had to repeatedly consider higher order expansions in terms of Green's functions coming from simple random walk, which led to serious problems, for example in Sections 4.3 and 4.4 in [6].

The reason for developing a new approach was twofold: On the one hand, it seemed difficult to fix these problems ad hoc. On the other hand, we aimed at establishing a solid basis for future work on this topic, in particular in the direction of a central limit theorem. Further new points of this work can be summarized as follows.

- We give either new proofs of the statements in [6] or we revise the old ones. For example, the proofs leading to the main results on the exit measures in Sections 5 and 6 are based on our new techniques. These include the bounds on Green's functions, the use of parametrized coarse graining schemes and the concept of goodified environments, which goes back to [6] and is further elaborated here.

- The appendix is completely rewritten. In this part, the main corrections concern the proof of the key Lemma 3.2 (Lemma 3.4 in [6]), where different case had to be considered. Also, we provide a lower bound on exit probabilities (Lemma 3.2 (iii)), which was already implicitly used in [6], but not proved.

- We obtain local estimates for the exit measures (Theorem 1.2). The global estimates in total variation distance are extended to starting points $|x| \leq L/5$.

- The results on the exit distributions are used to control the mean sojourn time of the RWRE in balls, under an extra assumption on $\mathbb{P}$.

To improve readability, we overview the main steps of this work in Section 1.5.

0.4 A brief history

The literature on random walks in random environment is vast, and we do by no means intend to give a full overview here. Instead, we point at some cornerstones and focus on results which are relevant for our particular model. For a more detailed survey, the reader is invited to consult the lecture notes of Sznitman [30], [32] and Zeitouni [38], [39], and also the overview article of Bogachev [7].

Recall the general model defined at the very beginning under "General description". We additionally assume that the environment is *uniformly elliptic*, i.e. there exists $\kappa > 0$ such that $\mathbb{P}$-almost surely, $\omega_x(e) \geq \kappa$ for all $x, e \in \mathbb{Z}^d$, $|e| = 1$. Note that in the perturbative regime, this is automatically true.

The natural condition of uniform ellipticity can sometimes be relaxed to mere ellipticity $\omega_x(e) > 0$ for $x, e \in \mathbb{Z}^d$, $|e| = 1$. Also, it often suffices to require $\mathbb{P}$ to be stationary and ergodic instead of being "i.i.d.".

Dimension $d = 1$

Early interest in models of RWRE can be traced back to the 60's in the context of biochemistry, where they were used as a toy model for DNA replication, cf. Chernov [9] and Temkin [35]. Solomon [27] started a rigorous mathematical analysis in dimension $d = 1$. He proved that if

$$\mathbb{E}\left[\log \rho\right] \neq 0, \quad \text{where } \rho = \omega_0(-1)/\omega_0(1),$$

then the RWRE is P-almost surely transient, whereas in the case $\mathbb{E}\left[\log \rho\right] = 0$, the walk is P-a.s. recurrent. Further, he obtained almost sure existence of the limit speed $v = \lim_{n\to\infty} X_n/n$,

$$v = \begin{cases} \frac{1-\mathbb{E}[\rho]}{1+\mathbb{E}[\rho]} & \text{if } \mathbb{E}[\rho] < 1 \\ \frac{1-\mathbb{E}[\rho^{-1}]}{1+\mathbb{E}[\rho^{-1}]} & \text{if } \mathbb{E}[\rho^{-1}] < 1 \\ 0 & \text{otherwise} \end{cases}.$$

His results already reveal some surprising features of the model. For example, it can happen that $v = 0$, but nonetheless the RWRE is transient (note that this is impossible for a Markov chain with stationary increments, according to Kesten [18]). Also, if $\overline{v} = \mathbb{E}[\omega_0(1) - \omega_0(-1)]$ denotes the mean local drift, it is possible that $|v| < |\overline{v}|$. Such slowdown effects, caused by traps reflecting impurities in the medium, come again to light in limit theorems for the RWRE under both the quenched and the annealed measure. In [20], Kesten, Kozlov and Spitzer proved that in the transient case under the annealed law, both diffusive and sub-diffusive behavior can occur, depending on a critical exponent connected to hitting times. However, the strongest form of sub-diffusivity appears in the recurrent case with non-degenerate site distribution μ, for which Sinai [26] proved that after n steps, the RWRE is typically at distance of order only $(\log n)^2$ away from the starting point. His analysis shows that the walk spends most of the time at the bottom of certain valleys. The limit law of $X_n/(\log n)^2$ is given by the distribution of a functional of Brownian motion, cf. Kesten [19] and Golosov [14]. Let us finally mention that slowdown phenomena also show up when studying probabilities of atypical events like large deviations, see e.g. [15], [11], [13].

Dimensions $d \geq 2$

While the one-dimensional picture is quite complete, many questions remain open in higher dimensions, including a classification into recurrent/transient behavior, existence of a limit speed and invariance principles. The main difficulties come from the non-Markovian character under the annealed measure and the fact that the RWRE is irreversible under the quenched measure as soon as $d \geq 2$.

Let us illustrate one prominent open problem, the directional zero-one law. For an element l from the unit sphere $\mathbb{S}^{d-1}$, denote the event that the RWRE is transient in direction l by

$$A_l = \left\{ \lim_{n\to\infty} X_n \cdot l = \infty \right\}.$$

Kalikow proved in [17] that $P(A_l \cup A_{-l}) \in \{0,1\}$. Is it also true that $P(A_l) \in \{0,1\}$? The answer is affirmative in dimension $d = 1, 2$ ([27] for $d = 1$, Merkl and Zerner [25] for $d = 2$), but unknown for higher dimensions. It is known that a limit speed $v \in \mathbb{R}^d$ (possibly zero) exists if $P(A_l) \in \{0,1\}$ for every $l \in \mathbb{S}^{d-1}$, cf. Sznitman and Zerner [34].

Much progress has been made in characterizing models which exhibit ballistic behavior, that is when the limit velocity v is an almost sure constant vector different from zero. Here Sznitman's conditions (T_γ), $\gamma \in (0,1]$, give a criterion for ballisticity and lead to an invariance principle under the annealed measure P, see Sznitman [28], [29] and also his lecture notes [32]. When $d \geq 4$ and the disorder is small, a quenched invariance principle has been shown by Bolthausen and Sznitman [4]. A stronger ballisticity condition was given earlier by Kalikow [17]. However, as examples in Sznitman [31] demonstrate, Kalikow's condition does not completely describe ballistic behavior in dimensions $d \geq 3$. A handy and complete characterization of ballisticity has still to be found. For recent developments, see the work of Berger [1] and Berger, Drewitz, Ramírez [2]. In [2], it is conjectured that in dimensions $d \geq 2$, a RWRE which is transient in all directions l out of an open subset $U \subset \mathbb{S}^{d-1}$ is ballistic (for an i.i.d uniformly elliptic environment).

Turning to ballistic behavior in the perturbative regime, Sznitman shows in [31] that for $0 < \eta < 5/2$ in dimension $d = 3$ or for $0 < \eta < 3$ in dimensions $d \geq 4$, there exists $\varepsilon_0 = \varepsilon_0(d,\eta)$ such that if $\mathbf{A0}(\varepsilon)$ is fulfilled for some $\varepsilon \leq \varepsilon_0$ and the mean local drift under the static measure satisfies

$$\mathbb{E}[d(0,\omega)\cdot e_1] > \begin{cases} \varepsilon^{5/2-\eta} & \text{if } d = 3 \\ \varepsilon^{3-\eta} & \text{if } d \geq 4 \end{cases}, \qquad \text{where } d(0,\omega) = \sum_{|e|=1} e\,\omega_0(e),$$

then the RWRE is ballistic in direction e_1, i.e. $v \cdot e_1 \neq 0$. Moreover, a functional limit theorem holds under P. In [5], Bolthausen, Sznitman and Zeitouni consider RWRE in dimensions $d \geq 6$ where the projection onto at least five components behaves as simple random walk. Among other things, examples are constructed under $\mathbf{A0}(\varepsilon)$ for which $\mathbb{E}[d(0,\omega)] \neq 0$, but $v = 0$ ($d \geq 7$), and a quenched invariance principle is proved when $d \geq 15$. On the other hand, it can happen that $\mathbb{E}[d(0,\omega)] = 0$ but $v \neq 0$. As a further remarkable result of [5], it can even happen that $0 \neq v = -c\mathbb{E}[d(0,\omega)]$ for some $c > 0$, which exemplifies that the environment acts on the path of the walk in a highly nontrivial way. Large deviations of X_n/n are studied in Varadhan [36].

Concerning non-ballistic behavior, much is known for the class of *balanced* RWRE when $\mathbb{P}(\omega_0(e_i) = \omega_0(-e_i)) = 1$ for all $i = 1, \ldots, d$. One first notices that the walk is a martingale, which readily leads to limit speed zero. Employing the method of environment viewed from the particle, Lawler proves in [22] that for $\mathbb{P}$-almost all ω, $X_{\lfloor n\cdot \rfloor}/\sqrt{n}$ converges in $\mathrm{P}_{0,\omega}$-distribution to a non-degenerate Brownian motion with diagonal covariance matrix. Moreover, the RWRE is recurrent in dimension $d = 2$ and transient when $d \geq 3$, see [38]. Recently, within the i.i.d. setting, diffusive behavior has been shown in the mere elliptic case by Guo and Zeitouni [16] and in the non-elliptic case by Berger and Deuschel [3].

Our study of random walks in random environment in the perturbative regime under the isotropy condition **A1** aims at a quenched central limit theorem, showing that in dimensions $d \geq 3$, the RWRE is asymptotically Gaussian, on $\mathbb{P}$-almost all environments ω. Such an invariance principle has already been shown by Bricmont and Kupiainen [8], who introduced condition **A1**. However, it is of interest to find a self-contained new proof. A continuous counterpart of this model, isotropic diffusions in a random environment which are small perturbations of Brownian motion, has been investigated by Sznitman and Zeitouni in [33]. They prove transience and a full quenched invariance principle in dimensions $d \geq 3$.

0.5 Open problems for our model and ongoing work

As we already pointed out above, with respect to a central limit theorem one still needs to find ways to handle large times, which are in a certain sense excluded by Assumption **A2**. In this direction, a more complete picture of exit laws could prove helpful, including sharper estimates for the appearance of balls with an atypical exit measure. A further task is to combine space and time estimates in the right way.

In the direction of a fully perturbative theory it would be desirable to replace the isometry condition **A1** by the requirement that μ is just invariant under reflections mapping a unit vector to its inverse. Then the RWRE exit law from a ball should be close to that of some d-dimensional symmetric random walk. The relaxed condition on μ would, for example, include the class of walks that are balanced in one coordinate direction, where time can be controlled much easier. This is work in progress.

Quite recently, Bolthausen and Zeitouni resumed working on the case of small isotropic perturbations in dimension $d = 2$. One expects diffusive behavior as in dimensions $d \geq 3$, but there is no rigorous result yet. In principle, one might try to follow a similar multi-scale approach for the exit measures as it is presented below. But the same perturbation argument shows that unlike dimensions $d \geq 3$, the disorder does not contract in leading order. Therefore, one has to look closer at higher order terms. While for $d \geq 3$, the nonlinear terms in the perturbation expansion for the Green's function can be estimated in a somewhat crude way once the right scales are found, it seems that in dimension $d = 2$, at least terms up to order three have to be carefully taken into account.

1 Basic notation and main results

1.1 Basic notation

Our purpose here is to cover the most relevant notation which will be used throughout this text. Further notation will be introduced later on when needed.

Sets and distances

We have $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ and $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. For a set A, its complement is denoted by A^c. If $A \subset \mathbb{R}^d$ is measurable and non-discrete, we write $|A|$ for its d-dimensional Lebesgue measure. Sometimes, $|A|$ denotes the surface measure instead, but this will be clear from the context. If $A \subset \mathbb{Z}^d$, then $|A|$ denotes its cardinality.

For $x \in \mathbb{R}^d$, $|x|$ is the Euclidean norm. If $A, B \subset \mathbb{R}^d$, we set $\mathrm{d}(A, B) = \inf\{|x - y| : x \in A,\ y \in B\}$ and $\mathrm{diam}(A) = \sup\{|x - y| : x, y \in A\}$. Given $L > 0$, let $V_L = \{x \in \mathbb{Z}^d : |x| \leq L\}$, and for $x \in \mathbb{Z}^d$, $V_L(x) = V_L + x$. For Euclidean balls in $\mathbb{R}^d$ we write $C_L = \{x \in \mathbb{R}^d : |x| < L\}$ and for $x \in \mathbb{R}^d$, $C_L(x) = x + C_L$.

If $V \subset \mathbb{Z}^d$, then $\partial V = \{x \in V^c \cap \mathbb{Z}^d : \mathrm{d}(\{x\}, V) = 1\}$ is the outer boundary, while in the case of a non-discrete set $V \subset \mathbb{R}^d$, ∂V stands for the usual topological boundary of V and $\overline{V}$ for its closure. For $x \in \overline{C}_L$, we set $\mathrm{d}_L(x) = L - |x|$. Finally, for $0 \leq a < b \leq L$, the "shell" is defined by

$$\mathrm{Sh}_L(a, b) = \{x \in V_L : a \leq \mathrm{d}_L(x) < b\}, \quad \mathrm{Sh}_L(b) = \mathrm{Sh}_L(0, b).$$

Functions

If a, b are two real numbers, we set $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$. The largest integer not greater than a is denoted by $\lfloor a \rfloor$. As usual, set $1/0 = \infty$. For us, log is the logarithm to the base e. For $x, z \in \mathbb{R}^d$, the Delta function $\delta_x(z)$ is defined to be equals one for $z = x$ and zero otherwise. If $V \subset \mathbb{Z}^d$ is a set, then δ_V is the probability distribution on the subsets of $\mathbb{Z}^d$ satisfying $\delta_V(V') = 1$ if $V' = V$ and zero otherwise.

Given two functions $F, G : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$, we write FG for the (matrix) product $FG(x, y) = \sum_{u \in \mathbb{Z}^d} F(x, u)G(u, y)$, provided the right hand side is absolutely summable. F^k is the kth power defined in this way, and $F^0(x, y) = \delta_x(y)$. F can also operate on functions $f : \mathbb{Z}^d \to \mathbb{R}$ from the left via $Ff(x) = \sum_{y \in \mathbb{Z}^d} F(x, y)f(y)$.

We use the symbol 1_W for the indicator function of the set W. By an abuse of notation, 1_W will also denote the kernel $(x, y) \mapsto 1_W(x)\delta_x(y)$. If $f : \mathbb{Z}^d \to \mathbb{R}$, $||f||_1 = \sum_{x \in \mathbb{Z}^d} |f(x)| \in [0, \infty]$ is its L^1-norm. When $\nu : \mathbb{Z}^d \to \mathbb{R}$ is a (signed) measure, $||\nu||_1$ is its total variation norm.

Let $U \subset \mathbb{R}^d$ be a bounded open set, and let $k \in \mathbb{N}$. For a k-times continuously differentiable function $f : U \to \mathbb{R}$, that is $f \in C^k(U)$, we define for $i = 0, 1, \ldots, k$,

$$\left|\left|D^i f\right|\right|_U = \sup_{|\beta| = i} \sup_U \left| \frac{\partial^i}{\partial x_1^{\beta_1} \cdots \partial x_d^{\beta_d}} f \right|,$$

where the first supremum is over all multi-indices $\beta = (\beta_1, \ldots, \beta_d)$, $\beta_j \in \mathbb{N}$, with $|\beta| = \sum_{j=1}^d \beta_j$. Let $L > 0$. Putting $U_L = \{x \in \mathbb{R}^d : L/2 < |x| < 2L\}$, we define

$$\mathcal{M}_L = \left\{\psi : U_L \to (L/10, 5L),\ \psi \in C^4(U_L), \left|\left|D^i\psi\right|\right|_{U_L} \leq 10 \text{ for } i = 1, \ldots, 4\right\}.$$

We will mostly interpret functions $\psi \in \mathcal{M}_L$ as maps from $U_L \cap \mathbb{Z}^d \subset \mathbb{R}^d$. A typical function we have in mind is the constant function $\psi \equiv L$.

Transition probabilities and exit distributions

Given (not necessarily nearest neighbor) transition probabilities $p = (p(x,y))_{x,y\in\mathbb{Z}^d}$, we write $\mathrm{P}_{x,p}$ for the law of the canonical Markov chain $(X_n)_{n\geq 0}$ on $((\mathbb{Z}^d)^{\mathbb{N}}, \mathcal{G})$, $\mathcal{G}$ the σ-algebra generated by cylinder functions, with transition probabilities p and starting point $X_0 = x$ $\mathrm{P}_{x,p}$ -a.s. The expectation with respect to $\mathrm{P}_{x,p}$ is denoted by $\mathrm{E}_{x,p}$. We will often consider the simple random walk kernel $p^{\mathrm{RW}}(x, x \pm e_i) = 1/(2d)$.

If $V \subset \mathbb{Z}^d$, we denote by $\tau_V = \inf\{n \geq 0 : X_n \notin V\}$ the first exit time from V, with $\inf \emptyset = \infty$, whereas $T_V = \tau_{V^c}$ is the first hitting time of V. Given $x, z \in \mathbb{Z}^d$ and p, V as above, we define

$$\mathrm{ex}_V(x, z;\, p) = \mathrm{P}_{x,p}\left(\tau_V = z\right).$$

Notice that for $x \in V^c$, $\mathrm{ex}_V(x, z; p) = \delta_x(z)$. For simple random walk, we write

$$\pi_V(x, z) = \mathrm{ex}_V\left(x, z; p^{\mathrm{RW}}\right).$$

Given $\omega \in \Omega$, we set

$$\Pi_V(x, z) = \mathrm{ex}_V(x, z; p_\omega).$$

Here, Π_V should be understood as a *random* exit distribution, but we suppress ω in the notation.

Coarse grained random walks

In order to transfer information about both exit measures and sojourn times from one scale to the next, we work with coarse graining schemes.

Fix once for all a probability density $\varphi \in C^\infty(\mathbb{R}_+, \mathbb{R}_+)$ with compact support in $(1, 2)$. Given a nonempty subset $W \subset \mathbb{Z}^d$, $x \in W$ and $m_x > 0$, the image measure of the rescaled density $(1/m_x)\varphi(t/m_x)\mathrm{d}t$ under the mapping $t \mapsto V_t(x) \cap W$ defines a probability distribution on (finite) sets containing x. If $\psi = (m_x)_{x\in W}$ is a field of positive numbers, we obtain in this way a collection of probability distributions indexed by $x \in W$, a *coarse graining scheme* on W.

Now if $p = (p(x,y))_{x\in W, y\in\mathbb{Z}^d}$ is a collection of transition probabilities on W, we define the coarse grained transitions belonging to (ψ, p) by

$$p_\psi^{\mathrm{CG}}(x, \cdot) = \frac{1}{m_x}\int_{\mathbb{R}_+} \varphi\left(\frac{t}{m_x}\right) \mathrm{ex}_{V_t(x)\cap W}(x, \cdot; p)\mathrm{d}t, \quad x \in W. \tag{1}$$

If $p = p^{\mathrm{RW}}$, we write $\hat{\pi}_\psi$ instead of p_ψ^{CG}. Note that for every choice of W and ψ, $\hat{\pi}_\psi$ defines a probability kernel.

For the motion in the ball V_L, we use a particular field ψ, which we describe in Section 2.1. There, we will also introduce a coarse grained RWRE transition kernel.

Further notation and abbreviations

For simplicity, we set $\mathrm{P}_x = \mathrm{P}_{x,p^{\mathrm{RW}}}$, $\mathrm{E}_x = \mathrm{E}_{x,p^{\mathrm{RW}}}$. Given transition probabilities p_ω coming from an environment ω, we use the notation $\mathrm{P}_{x,\omega}$, $\mathrm{E}_{x,\omega}$. In order to avoid double indices, we usually write π_L instead of π_V, Π_L for Π_V and τ_L for τ_V if $V = V_L$ is the ball of radius L around zero.

Many of our quantities will be indexed by both L and r, where r is an additional parameter. While we always keep the indices in the statements, we normally drop both of them in the proofs. We will often use the abbreviations $\mathrm{d}(y, B)$ for $\mathrm{d}(\{y\}, B)$, T_x for $T_{\{x\}}$ and $\mathbb{P}\left(A;\, B\right)$ for $\mathbb{P}\left(A \cap B\right)$.

Some words about constants, O-notation and large L behavior

All our constants are positive. They only depend on the dimension $d \geq 3$ unless stated otherwise. In particular, they do *not* depend on L, on ω or on any point $x \in \mathbb{Z}^d$, and they are also independent of the parameter r which will be introduced in Section 2.

We use C and c for generic positive constants whose values can change in different expressions, even in the same line. If we use other constants like K, C_1, c_1, their values are fixed throughout the proofs. Lower-case constants usually indicate small (positive) values.

Given two functions f, g defined on some subset of $\mathbb{R}$, we write $f(t) = O\left(g(t)\right)$ if there exists a positive $C > 0$ and a real number t_0 such that $|f(t)| \leq C|g(t)|$ for $t \geq t_0$.

If a statement holds for "L large (enough)", this means that there exists $L_0 > 0$ depending only on the dimension such that the statement is true for all $L \geq L_0$. This applies analogously to the expressions "δ (or ε) small (enough)".

The reader should always keep in mind that we are interested in asymptotics when $L \to \infty$ and ε is a (arbitrarily) small positive constant. Even though some of our statements are valid only for large L and ε sufficiently small, we do not mention this every time.

1.2 Main results on exit laws

We still need some notation. For $x \in \mathbb{Z}^d$, $t > 0$ and $\psi : \partial V_t(x) \to (0, \infty)$ define

$$D_{t,\psi}(x) = \left|\left|\left(\Pi_{V_t(x)} - \pi_{V_t(x)}\right)\hat{\pi}_\psi(x, \cdot)\right|\right|_1,$$

$$D_t(x) = \left|\left|\left(\Pi_{V_t(x)} - \pi_{V_t(x)}\right)(x, \cdot)\right|\right|_1.$$

If $\psi \equiv m$ is constant, we write $D_{t,m}$ instead of $D_{t,\psi}$. We usually drop x from the notation if $x = 0$. Further, let

$$D^*_{t,\psi} = \sup_{x \in V_{t/5}} \left|\left|\left(\Pi_{V_t} - \pi_{V_t}\right)\hat{\pi}_\psi(x, \cdot)\right|\right|_1,$$

$$D_t^* = \sup_{x \in V_{t/5}} ||(\Pi_{V_t} - \pi_{V_t})(x, \cdot)||_1 .$$

With $\delta > 0$, we set for $i = 1, 2, 3$

$$b_i(L, \psi, \delta) = \mathbb{P}\left(\left\{(\log L)^{-9+9(i-1)/4} < D_{L,\psi}^* \leq (\log L)^{-9+9i/4}\right\} \cap \{D_L^* \leq \delta\}\right),$$

and

$$b_4(L, \psi, \delta) = \mathbb{P}\left(\left\{D_{L,\psi}^* > (\log L)^{-3+3/4}\right\} \cup \{D_L^* > \delta\}\right).$$

The following technical condition will play a key role.

Let $\delta > 0$ and $L_1 \geq 3$. We say that $\mathbf{C1}(\delta, L_1)$ holds if for all $3 \leq L \leq L_1$, all $\psi \in \mathcal{M}_L$, $i = 1, 2, 3, 4$,

$$b_i(L, \psi, \delta) \leq \frac{1}{4} \exp\left(-\left((3+i)/4\right)(\log L)^2\right).$$

Notice that if $\mathbf{C1}(\delta, L_1)$ is satisfied, then for any $3 \leq L \leq L_1$ and any $\psi \in \mathcal{M}_L$,

$$\mathbb{P}\left(\left\{D_{L,\psi}^* > (\log L)^{-9}\right\} \cup \{D_L^* > \delta\}\right) \leq \exp\left(-(\log L)^2\right).$$

We can now formulate our results. Proposition 1.1, Theorem 1.1 and Corollary 1.1 are in a similar form already present in [6]. See also our discussion in the introduction.

The main technical statement is

Proposition 1.1. *Assume* **A1**. *There exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0]$, there exists $\varepsilon_0 = \varepsilon_0(\delta) > 0$ with the following property: If $\varepsilon \leq \varepsilon_0$ and $\mathbf{A0}(\varepsilon)$ holds, then*

(i) *There exists $L_0 = L_0(\delta)$ such that for $L_1 \geq L_0$,*

$$\mathbf{C1}(\delta, L_1) \Rightarrow \mathbf{C1}\left(\delta, L_1(\log L_1)^2\right).$$

(ii) *There exist sequences l_n, $m_n \to \infty$ such that if $L_1 \geq l_n$ and $L_1 \leq L \leq L_1(\log L_1)^2$, $m \geq m_n$,*

$$\mathbf{C1}(\delta, L_1) \Rightarrow \left(\mathbb{P}\left(D_{L,m}^* > 1/n\right) \leq \exp\left(-(\log L)^2\right)\right).$$

As a direct consequence, we get

Theorem 1.1 ($d \geq 3$)**.** *Assume* **A1**.

(i) *There exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0]$, there exists $\varepsilon_0 = \varepsilon_0(\delta) > 0$ with the following property: If $\varepsilon \leq \varepsilon_0$ and $\mathbf{A0}(\varepsilon)$ holds, then for all $L \geq 1$,*

$$\mathbb{P}\left(D_L^* > \delta\right) \leq \exp\left(-(\log L)^2\right).$$

(ii) *There exists $\varepsilon_0 > 0$ such that if $\mathbf{A0}(\varepsilon)$ is satisfied for some $\varepsilon \leq \varepsilon_0$, then for any $\eta > 0$, we can find L_η and a smoothing radius m_η such that for $m \geq m_\eta$, $L \geq L_\eta$,*

$$\mathbb{P}\left(D_{L,m}^* > \eta\right) \leq \exp\left(-(\log L)^2\right).$$

Remark 1.1. (i) In particular, part (i) of Proposition 1.1 tells us that if $\delta \leq \delta_0$, then $\mathbf{C1}(\delta, L)$ holds for all large L, provided $\mathbf{A0}(\varepsilon)$ is fulfilled for ε small enough, depending only on δ (and the dimension). This follows immediately from the fact that given any $\delta > 0$ and any $L_1 \geq 3$, we can always find $\varepsilon > 0$ such that $\mathbf{A0}(\varepsilon)$ implies $\mathbf{C1}(\delta, L_1)$.
(ii) As an easy consequence of part (ii) of the theorem, if one increases the smoothing scale with L, i.e. if $m = m_L \uparrow \infty$ (arbitrary slowly) as $L \to \infty$, then

$$D^*_{L,m_L} \to 0 \quad \mathbb{P}\text{-almost surely.}$$

(iii) One could define the smoothing kernel $\hat{\pi}_\psi$ differently. However, our particular form is useful for the induction procedure.

Our methods allow us to compare the exit measures in a more local way. For positive t and $z \in \partial V_L$, let $W_t(z) = V_t(z) \cap \partial V_L$. Then $W_t(z)$ contains on the order of t^{d-1} points. The center $z \in \partial V_L$ will play no particular role, so we drop it from the notation. If we choose our parameters according to Theorem 1.1 (i), we have good control over $\Pi_L(x, W_t)$ in terms of $\pi_L(x, W_t)$, provided x has a distance of order L from the boundary and t is sufficiently large. For the statement of the following theorem, we pick $\delta \in (0, \delta_0]$ and $L_0(\delta)$ according to Proposition 1.1, and choose the perturbation $\varepsilon \leq \varepsilon_0$ small enough such that $\mathbf{A0}(\varepsilon)$ implies $\mathbf{C1}(\delta, L_0)$ (and then $\mathbf{C1}(\delta, L)$ for all $L \geq L_0$, according to the proposition).

Theorem 1.2. *Assume* $\mathbf{A1}$. *In the setting just described, if* $\mathbf{A0}(\varepsilon)$ *is fulfilled, then for* $L \geq L_0$, *there exists an event* $A_L \in \mathcal{F}$ *with* $\mathbb{P}(A_L^c) \leq \exp(-(1/2)(\log L)^2)$ *such that on* A_L, *the following holds true. If* $0 < \eta < 1$ *and* $x \in V_{\eta L}$, *then*

(i) *For* $t \geq L/(\log L)^{15}$ *and every set* W_t *as above, there exists* $C = C(\eta)$ *with*

$$\Pi_L(x, W_t) \leq C \pi_L(x, W_t).$$

(ii) *For* $t \geq L/(\log L)^6$,

$$\Pi_L(x, W_t) = \pi_L(x, W_t)\left(1 + O\left((\log L)^{-5/2}\right)\right).$$

Here, the constant in the O*-notation depends only on* d *and* η.

From Proposition 1.1, we also obtain transience of the RWRE.

Corollary 1.1 (Transience)**.** *Assume* $\mathbf{A1}$. *There exist* ε_0, $\rho > 0$ *such that if* $\mathbf{A0}(\varepsilon)$ *is satisfied for some* $\varepsilon \leq \varepsilon_0$, *then for* $\mathbb{P}$*-almost all* $\omega \in \Omega$, *there exists* $m_0 = m_0(\omega) \in \mathbb{N}$ *with the following property: For integers* $m \geq m_0$ *and* $k \geq 1$,

$$\sup_{x : |x| \geq \rho^{m+k}} \mathrm{P}_{x,\omega}\left(T_{V_{\rho^m}} < \infty\right) \leq (2/3)^k. \tag{2}$$

In particular, the RWRE $(X_n)_{n \geq 0}$ *is transient.*

1.3 Main results on mean sojourn times

For the times, we propagate a condition similar to $\mathbf{C1}(\delta, L)$. In this regard, we first introduce a monotone increasing function which will limit the normalized mean sojourn time in the ball. Let $0 < \eta < 1$, and define $f_\eta : \mathbb{R}_+ \to \mathbb{R}_+$ by setting

$$f_\eta(L) = \frac{\eta}{3} \sum_{k=1}^{\lceil \log L \rceil} k^{-3/2}.$$

Note that $\eta/3 \leq f_\eta(L) < \eta$ and therefore $\lim_{\eta \downarrow 0} \lim_{L\to\infty} f_\eta(L) = 0$.

Recall that E_0 is the expectation with respect to simple random walk starting at the origin. We say that $\mathbf{C2}(\eta, L_1)$ holds, if for all $3 \leq L \leq L_1$,

$$\mathbb{P}\left(\mathrm{E}_{0,\omega}\left[\tau_L\right] \notin \left[1 - f_\eta(L),\, 1 + f_\eta(L)\right] \cdot \mathrm{E}_0\left[\tau_L\right]\right) \leq L^{-6d}.$$

Our main technical result is

Proposition 1.2. *Assume* **A1** *and* **A2**, *and let* $0 < \eta < 1$. *There exists* $\varepsilon_0 = \varepsilon_0(\eta) > 0$ *with the following property: If* $\varepsilon \leq \varepsilon_0$ *and* $\mathbf{A0}(\varepsilon)$ *holds, then*

(i) *There exists* $L_0 = L_0(\eta) > 0$ *such that for* $L_1 \geq L_0$,

$$\mathbf{C2}(\eta, L_1) \Rightarrow \mathbf{C2}(\eta, L_1(\log L_1)^2).$$

(ii)

$$\lim_{L\to\infty} L^d\, \mathbb{P}\left(\sup_{x:|x|\leq L^3} \sup_{y \in V_L(x)} \mathrm{E}_{y,\omega}\left[\tau_{V_L(x)}\right] \notin [1-\eta, 1+\eta] \cdot L^2\right) = 0.$$

By Borel-Cantelli and the Markov property, we immediately have

Corollary 1.2 (Quenched moments)**.** *In the framework of Proposition 1.2, for* $k \in \mathbb{N}$ *and* $\mathbb{P}$*-almost all* $\omega \in \Omega$,

$$\lim_{L\to\infty} \left(\sup_{x:|x|\leq L^3} \sup_{y \in V_L(x)} \mathrm{E}_{y,\omega}\left[\tau^k_{V_L(x)}\right] / L^{2k}\right) \leq 2^k k!\,.$$

The bounds on the quenched moments for $k = 2$ are useful to prove

Theorem 1.3. *Assume* **A1** *and* **A2**. *Given* $0 < \eta < 1$, *one can find* $\varepsilon_0 = \varepsilon_0(\eta) > 0$ *such that if* $\mathbf{A0}(\varepsilon)$ *is satisfied for some* $\varepsilon \leq \varepsilon_0$, *then the following holds: There exist* D_1, $D_2 \in [1-\eta, 1+\eta]$ *such that for* $\mathbb{P}$*-almost all* ω,

$$\begin{aligned}
\liminf_{L\to\infty} \left(\sup_{x:|x|\leq L^3} \sup_{y \in V_L(x)} \mathrm{E}_{y,\omega}\left[\tau_{V_L(x)}\right] / L^2\right) &= D_1,\\
\limsup_{L\to\infty} \left(\sup_{x:|x|\leq L^3} \sup_{y \in V_L(x)} \mathrm{E}_{y,\omega}\left[\tau_{V_L(x)}\right] / L^2\right) &= D_2.
\end{aligned}$$

Remark 1.2. (i) Given η and L_1, we can always guarantee (by making ε smaller if necessary) that $\mathbf{A0}(\varepsilon)$ implies $\mathbf{C2}(\eta, L_1)$.
(ii) The factor L^{-6d} in the definition of condition $\mathbf{C2}(\eta, L_1)$, the factor L^d and also the choice of L^3 inside the probability in the statement of Proposition 1.2 (ii) are connected to Assumption **A2**. If, for instance, one could prove that for some $\alpha > 1$ and large L,

$$\mathbb{P}\left(\mathrm{E}_{0,\omega}\left[\tau_L\right] > (\log L)^4 L^2\right) \leq \exp\left(-(\log L)^{\alpha}\right),$$

then Proposition 1.2 (ii) would hold with L^d replaced by L^r for every $r \in \mathbb{N}$.
(iii) In the last theorem, we strongly believe that $D_1 = D_2$.

1.4 Perturbation expansion

Our approach of comparing the behavior of the RWRE in space and time with that of simple random walk is based on a perturbation argument. Namely, the resolvent equation allows us to express Green's functions of the RWRE in terms of ordinary Green's functions. More generally, let $p = (p(x,y))_{x,y \in \mathbb{Z}^d}$ be a family of finite range transition probabilities on $\mathbb{Z}^d$, and let $V \subset \mathbb{Z}^d$ be a finite set. The corresponding Green's kernel or Green's function for V is defined by

$$g_V(p)(x,y) = \sum_{k=0}^{\infty} \left(1_V p\right)^k (x,y).$$

The connection with the exit measure is given by the fact that for $z \notin V$, we have

$$g_V(p)(\cdot, z) = \mathrm{ex}_V(\cdot, z; p).$$

Now write g for $g_V(p)$ and let P be another transition kernel with corresponding Green's function G for V. With $\Delta = 1_V\left(P - p\right)$, we have by the resolvent equation

$$G - g = g\Delta G = G\Delta g. \tag{3}$$

In order to get rid of G on the right hand side, we iterate (3) and obtain

$$G - g = \sum_{k=1}^{\infty} \left(g\Delta\right)^k g, \tag{4}$$

provided the infinite series converges, which will always be the case in our setting, due to $\mathbf{A0}(\varepsilon)$ and V being finite. A modification of (4) turns out to be particularly useful. Note that by (4),

$$G = g \sum_{k=0}^{\infty} \left(\Delta g\right)^k.$$

Replacing the rightmost g by $g(x,\cdot) = \delta_x(\cdot) + 1_V p g(x,\cdot)$ and reordering terms, we arrive at

$$G = g \sum_{m=0}^{\infty} \left(Rg\right)^m \sum_{k=0}^{\infty} \Delta^k, \tag{5}$$

where $R = \sum_{k=1}^{\infty} \Delta^k p$.

1.5 A short reading guide

The key idea behind our results on exit laws from V_L is to compare the RWRE exit measure with that of simple random walk by means of the perturbation expansion

$$\Pi_L - \pi_L = \hat{G}1_{V_L}(\hat{\Pi} - \hat{\pi})\pi_L.$$

Here, $\hat{\Pi}$ is a coarse grained RWRE transition kernel inside V_L, $\hat{\pi}$ is a coarse grained simple random walk kernel, and $\hat{G} = \hat{G}(\hat{\Pi})$ is the Green's function associated to $\hat{\Pi}$.

Our coarse grained transition kernels are given by exit distributions from smaller balls inside V_L, and we obtain our results by transfering inductively estimates on smaller scales to scale L. The coarse graining schemes defined in Section 2 determine the radii of the smaller balls. In the bulk of V_L, we choose the radius $s_L = L/(\log L)^3$, but we refine the radii when approaching the boundary. Our schemes are parametrized by a real number r, which determines the distance to the boundary ∂V_L at which the refinement stops. We choose r equal to $r_L = L/(\log L)^{15}$ for the estimates involving a global smoothing, and equals a large constant for the non- or locally smoothed estimates.

Besides the coarse graining schemes, Section 2 introduces the concept of "good" and "bad" points and so-called goodified Green's functions. Roughly speaking, we call a point $x \in V_L$ *good* if the exit measure on such a smaller ball around x is close to the exit measure of simple random walk, in both a smoothed and non-smoothed way. If inside V_L all points are good, then the estimates on smaller balls can be transferred to a (globally smoothed) estimate on the larger ball V_L (Lemma 5.2), using some averaging argument and an exponential inequality.

But *bad* points can appear, and in fact we have to distinguish four different levels of badness (Section 2.3). When bad points are present, it is convenient to "goodify" the environment, that is to replace bad points by good ones. This important concept is first explained in Section 2 and then further developed in Section 4. However, for the globally smoothed estimate, due to the additional smoothing step we only have to deal with the case where all bad points are enclosed in a comparably small region - two or more such regions are too unlikely (Lemma 2.1). Some special care is required for the worst class of bad points in the interior of the ball. For environments containing such points, we slightly modify the coarse graining scheme inside V_L, as described in Section 4.4. In Lemma 5.3, we prove the smoothed estimates on environments with bad points and show that the degree of badness decreases by one from one scale to the next.

Concerning exit measures where no or only a local last smoothing step is added (Section 6, Lemmata 6.1 and 6.2, respectively), bad points near the boundary of V_L are much more delicate to handle, since we have to take into account several possibly bad regions. However, they do not occur too frequently (Lemma 2.2) and can be controlled by capacity arguments.

All these estimates require precise bounds on coarse grained Green's functions, which are developed in Section 4. Basically, we show that on environments with no bad points, the coarse grained RWRE Green's function for the ball can be estimated from above by the analogous quantity coming from simple random walk.

Section 3 is devoted to technical bounds on hitting probabilities of both simple random walk and Brownian motion, and to difference estimates of smoothed exit measures. The reason for working sometimes with Brownian motion instead of a random walk is of technical nature - some estimates are easier to prove for the former, as for example Lemma 3.5 (iii). They can then be transferred to random walks via coupling arguments provided in the appendix.

The statements from Sections 5 and 5 are finally used in Section 7 to prove the main results on exit measures, including the proof of transience of the RWRE.

The object of interest in Section 8 is the mean sojourn time of the RWRE in the ball V_L. Employing the Markov property, we represent this quantity as a convolution of a coarse grained RWRE Green's function $\hat{G}$ and mean sojourn times in smaller balls $\Lambda_L(y)$,

$$\mathrm{E}_{x,\omega}\left[\tau_L\right] = \sum_{y\in V_L} \hat{G}(x,y)\Lambda_L(y).$$

Again, multiscale analysis is used to transport time estimates on a smaller to a bigger scale. It turns out that we need control over space and time on the next two lower levels. This requires a stronger notion of good and bad points concerning both spatial and temporal behavior. In Section 9, we prove our main results on mean sojourn times.

Finally, in the appendix we prove the main statements of Section 3, as well as a local central limit theorem for the coarse grained simple random walk.

2 Coarse graining schemes and notion of badness

The purpose of this section is to introduce coarse graining schemes in the ball as well as the concept of "good" and "bad" points. Also, we prove two estimates ensuring that we do not have to consider environments with bad points that are widely spread out in the ball or densely packed in the boundary region.

2.1 Coarse graining schemes in the ball

Once for all, define

$$s_L = \frac{L}{(\log L)^3} \quad \text{and} \quad r_L = \frac{L}{(\log L)^{15}}.$$

We will use particular coarse graining schemes indexed by a parameter r, which can either be a constant ≥ 100, but much smaller than r_L, or, in most of the cases, $r = r_L$. We fix a smooth function $h : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$h(x) = \begin{cases} x & \text{for } x \leq \frac{1}{2} \\ 1 & \text{for } x \geq 2 \end{cases},$$

such that h is strictly monotone and concave on $(1/2, 2)$, with first derivative bounded uniformly by 1. Define $h_{L,r} : \overline{C}_L \to \mathbb{R}_+$ by

$$h_{L,r}(x) = \frac{1}{20} \max\left\{ s_L h\left(\frac{\mathrm{d}_L(x)}{s_L}\right), r\right\}. \tag{6}$$

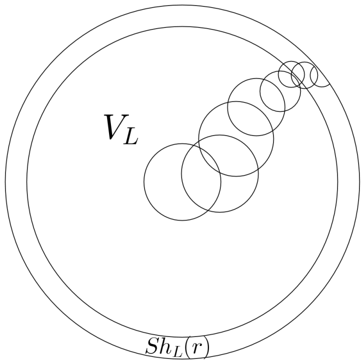

Figure 1: The coarse graining scheme in V_L. In the bulk $\{x \in V_L : d_L(x) \geq 2s_L\}$, the exit distributions are taken from balls of radii between $(1/20)s_L$ and $(1/10)s_L$. When entering $\mathrm{Sh}_L(2s_L)$, the coarse graining radii start to shrink, up to the boundary layer $\mathrm{Sh}_L(r)$, where the exit distributions are taken from intersected balls $V_t(x) \cap V_L$, $t \in [(1/20)r, (1/10)r]$.

Since we mostly work with $r = r_L$, we use the abbreviation $h_L = h_{L,r_L}$. We write $\hat{\Pi}_{L,r}$ for the coarse grained RWRE transition kernel associated to $(\psi = (h_{L,r}(x))_{x\in V_L}, p_\omega)$,

$$\hat{\Pi}_{L,r}(x,\cdot) = \frac{1}{h_{L,r}(x)} \int_{\mathbb{R}_+} \varphi\left(\frac{t}{h_{L,r}(x)}\right) \Pi_{V_t(x)\cap V_L}(x,\cdot)\mathrm{d}t,$$

and $\hat{\pi}_{L,r}$ for that coming from simple random walk, where Π is replaced by π. For convenience, we set $\hat{\Pi}_{L,r}(x,\cdot) = \hat{\pi}_{L,r}(x,\cdot) = \delta_x(\cdot)$ for $x \in \mathbb{Z}^d \backslash V_L$. Notice that by the strong Markov property, the exit measures from the ball V_L remain unchanged under these transition kernels, i.e.

$$\mathrm{ex}_{V_L}\left(x,\cdot;\hat{\Pi}_{L,r}\right) = \Pi_L(x,\cdot) \quad \text{and} \quad \mathrm{ex}_{V_L}\left(x,\cdot;\hat{\pi}_{L,r}\right) = \pi_L(x,\cdot).$$

We denote by $\hat{G}_{L,r}$ the (coarse grained) RWRE Green's function coming from $\hat{\Pi}_{L,r}$, and by $\hat{g}_{L,r}$ the Green's function from $\hat{\pi}_{L,r}$, everything in V_L.

Remark 2.1. (i) Later on, we will also work with slightly modified transition kernels $\breve{\Pi}$ and $\breve{\pi}$, which depend on the environment. We elaborate on this in Section 4.4.
(ii) Due to the lack of the last smoothing step outside V_L, we need to zoom in near the boundary in order to handle non-smoothed exit distributions in Section 6. The parameter r allows us to adjust the step size in the boundary region.
(ii) Note that for every choice of r,

$$h_{L,r}(x) = \begin{cases} \mathrm{d}_L(x)/20 & \text{for } x \in V_L \text{ with } r_L \leq \mathrm{d}_L(x) \leq s_L/2 \\ s_L/20 & \text{for } x \in V_L \text{ with } \mathrm{d}_L(x) \geq 2s_L \end{cases}.$$

2.2 Good and bad points

We shall partition the grid points inside V_L according to their influence on the exit behavior. We say that a point $x \in V_L$ is *good* (with respect to L, $\delta > 0$ and r, $100 \leq r \leq r_L$) if

- For all $t \in [h_{L,r}(x), 2h_{L,r}(x)]$, $\left|\left|(\Pi_{V_t(x)} - \pi_{V_t(x)})(x,\cdot)\right|\right|_1 \leq \delta$.

- If $\mathrm{d}_L(x) > 2r$, then additionally

$$\left|\left|(\hat{\Pi}_{L,r} - \hat{\pi}_{L,r})\hat{\pi}_{L,r}(x,\cdot)\right|\right|_1 \leq (\log h_{L,r}(x))^{-9}.$$

A point $x \in V_L$ which is not good is called *bad*. We denote by $\mathcal{B}_{L,r} = \mathcal{B}_{L,r}(\omega)$ the set of all bad points inside V_L and write $\mathcal{B}_L = \mathcal{B}_{L,r_L}$ for short. Furthermore, set $\mathcal{B}^{\partial}_{L,r} = \mathcal{B}_{L,r} \cap \mathrm{Sh}_L(r_L)$ and $\mathcal{B}^{\star}_{L,r} = \mathcal{B}_{L,r} \cup \mathcal{B}_L = \mathcal{B}^{\partial}_{L,r} \cup \mathcal{B}_L$. Of course, the set of bad points depends also on δ, but we do not indicate this.

Remark 2.2. (i) For the coarse graining scheme associated to $r = r_L$, we have by definition $\mathcal{B}^{\star}_{L,r_L} = \mathcal{B}_L$. When performing the estimates in Section 6, we work with constant r. In this case, $\mathcal{B}^{\star}_{L,r}$ can contain more points than $\mathcal{B}_L$.
(ii) Assume L large. If $x \in V_L$ with $\mathrm{d}_L(x) > 2r$, then the function $h_{L,r}(x + \cdot)$, defined in (6), lies in $\mathcal{M}_t$ for each $t \in [h_{L,r}(x), 2h_{L,r}(x)]$. Thus, for all $x \in V_L$, we can use $\mathbf{C1}(\delta, L_1)$ to control the event $\{x \in \mathcal{B}_{L,r}\}$, provided $2h_{L,r}(x) \leq L_1$. We make use of this in Lemma 2.1.

Goodified transition kernels

It is difficult to obtain estimates for the RWRE in the presence of bad points. For all environments, we therefore introduce "goodified" transition kernels $\hat{\Pi}^g_{L,r}$,

$$\hat{\Pi}^g_{L,r}(x,\cdot) = \begin{cases} \hat{\Pi}_{L,r}(x,\cdot) & \text{for } x \in V_L \backslash \mathcal{B}^{\star}_{L,r} \\ \hat{\pi}_{L,r}(x,\cdot) & \text{for } x \in \mathcal{B}^{\star}_{L,r} \end{cases}. \tag{7}$$

Furthermore, we write $\hat{G}^g_{L,r}$ for the corresponding (random) Green's function.

2.3 Bad regions in the case $r = r_L$

The following lemma shows that with high probability, all bad points with respect to $r = r_L$ are contained in a ball of radius $4h_L(x)$. Let

$$\mathcal{D}_L = \left\{V_{4h_L(x)}(x) : x \in V_L\right\}.$$

We will look at the events $\mathrm{OneBad}_L = \{\mathcal{B}_L \subset D \text{ for some } D \in \mathcal{D}_L\}$ and $\mathrm{ManyBad}_L = (\mathrm{OneBad}_L)^c$. It is also useful to define the set of *good* environments, $\mathrm{Good}_L = \{\mathcal{B}_L = \emptyset\} \subset \mathrm{OneBad}_L$.

Lemma 2.1. *For large L_1, $\mathbf{C1}(\delta, L_1)$ implies that for L with $L_1 \leq L \leq L_1(\log L_1)^2$,*

$$\mathbb{P}\left(\mathrm{ManyBad}_L\right) \leq \exp\left(-\frac{19}{10}(\log L)^2\right).$$

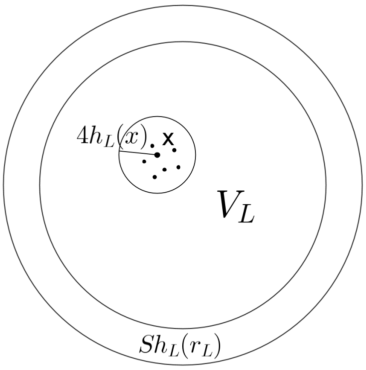

Figure 2: On environments $\omega \in \mathrm{OneBad}_L$, all bad points are enclosed in a ball $V_{4h_L(x)}(x)$.

Proof: Set $\Delta = 1_{V_L}(\hat{\Pi}_{L,r_L} - \hat{\pi}_{L,r_L})$, $\hat{\pi} = \hat{\pi}_{L,r_L}$. For all $x \in V_L$ with $\mathrm{d}_L(x) > 2r_L$, using $\frac{1}{20} r_L \le h_L(x) \le s_L \le L_1/2$,

$$\begin{aligned} \mathbb{P}(x \in \mathcal{B}_L) &= \mathbb{P}\left(\left\{||\Delta\hat{\pi}(x,\cdot)||_1 > (\log h_L(x))^{-9}\right\} \cup \left\{||\Delta(x,\cdot)||_1 > \delta\right\}\right) \\ &\le \mathbb{P}\left(\bigcup_{t\in[h_L(x),2h_L(x)]} \left\{D_{t,h_L}(x) > (\log h_L(x))^{-9}\right\} \cup \left\{D_t(x) > \delta\right\}\right) \\ &\le C s_L^d \exp\left(-\left(\log(r_L/20)\right)^2\right), \end{aligned}$$

and a similar estimate holds in the case $\mathrm{d}_L(x) \le 2r_L$. On the event $\mathrm{ManyBad}_L$, there exist $x, y \in \mathcal{B}_L$ with $|x-y| > 2h_L(x) + 2h_L(y)$. But for such x, y, the events $\{x \in \mathcal{B}_L\}$ and $\{y \in \mathcal{B}_L\}$ are independent, whence for L large

$$\mathbb{P}(\mathrm{ManyBad}_L) \le C L^{2d} s_L^{2d} \left[\exp\left(-(\log(r_L/20))^2\right)\right]^2 \le \exp\left(-(19/10)(\log L)^2\right).$$

□

The estimate is good enough for our inductive procedure, so we only have to deal with the case where all bad points are enclosed in a ball $D \in \mathcal{D}_L$. However, inside D we need to look closer at the degree of badness. We say that $\omega \in \mathrm{OneBad}_L$ is *bad on level* i, $i = 1, 2, 3$, if the following holds:

- For all $x \in V_L$, for all $t \in [h_L(x), 2h_L(x)]$, $\left|\left|(\Pi_{V_t(x)} - \pi_{V_t(x)})(x,\cdot)\right|\right|_1 \le \delta$.
- For all $x \in V_L$ with $\mathrm{d}_L(x) > 2r_L$, additionally

$$\left|\left|(\hat{\Pi}_{L,r_L} - \hat{\pi}_{L,r_L})\hat{\pi}_{L,r_L}(x,\cdot)\right|\right|_1 \le (\log h_L(x))^{-9+9i/4}.$$

- There exists $x \in \mathcal{B}_L(\omega)$ with $\mathrm{d}_L(x) > 2r_L$ such that

$$\left|\left|(\hat{\Pi}_{L,r_L} - \hat{\pi}_{L,r_L})\hat{\pi}_{L,r_L}(x,\cdot)\right|\right|_1 > (\log h_L(x))^{-9+9(i-1)/4}.$$

If $\omega \in \mathrm{OneBad}_L$ is neither bad on level $i = 1, 2, 3$ nor good, we call ω *bad on level* 4. In this case, $\mathcal{B}_L(\omega)$ contains "really bad" points. We write $\mathrm{OneBad}_L^{(i)} \subset \mathrm{OneBad}_L$ for the subset of all those ω which are bad on level $i = 1, 2, 3, 4$. Observe that

$$\mathrm{OneBad}_L = \mathrm{Good}_L \,\dot\cup \dot{\bigcup_{i=1}^{4}} \mathrm{OneBad}_L^{(i)}.$$

On Good_L, $\hat{\Pi}^g_{L,r_L} = \hat{\Pi}_{L,r_L}$ and therefore $\hat{G}^g_{L,r_L} = \hat{G}_{L,r_L}$.

2.4 Bad regions when r is a constant

When estimating the non-smoothed quantity D_L^*, we cannot stop the refinement of the coarse graining in the boundary region $\mathrm{Sh}_L(r_L)$. Instead, we will choose r as a (large) constant. However, now it is no longer true that essentially all bad points are contained in one single region $D \in \mathcal{D}_L$. For example, if $x \in V_L$ such that $\mathrm{d}_L(x)$ is of order $\log L$, we only have a bound of the form

$$\mathbb{P}\left(x \in \mathcal{B}_{L,r}\right) \leq \exp\left(-c(\log\log L)^2\right),$$

which is clearly not enough to get an estimate as in Lemma 2.1. We therefore choose a different strategy to handle bad points within $\mathrm{Sh}_L(r_L)$. We split the boundary region into layers of an appropriate size and use independence to show that with high probability, bad regions are rather sparse within those layers. Then the Green's function estimates of Corollary 4.1 will ensure that on such environments, there is a high chance to never hit points in $\mathcal{B}^\partial_{L,r}$ before leaving the ball.

To begin with the first part, fix r with $r \geq r_0 \geq 100$, where $r_0 = r_0(d)$ is a constant that will be chosen below. Let L be large enough such that $r < r_L$, and set $J_1 = J_1(L) = \left\lfloor \frac{\log(r_L/r)}{\log 2} \right\rfloor + 1$. We define layers $\Lambda_0 = \mathrm{Sh}_L(2r)$ and $\Lambda_j = \mathrm{Sh}_L(r2^j, r2^{j+1})$, $1 \leq j \leq J_1$. Then,

$$\mathrm{Sh}_L(2r_L) \subset \bigcup_{0 \leq j \leq J_1} \Lambda_j \subset \mathrm{Sh}_L(4r_L).$$

Let $j \in \mathbb{N}$. For $k \in \mathbb{Z}$, consider the interval $I_k^{(j)} = (kr2^j, (k+1)r2^j] \cap \mathbb{Z}$. We divide Λ_j into subsets by setting $D_{\mathbf{k}}^{(j)} = \Lambda_j \cap (I_{k_1} \times \ldots \times I_{k_d})$, where $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{Z}^d$. Denote by $\mathcal{Q}_{j,r}$ the set of these subsets which are not empty. Setting $N_{j,r} = |\mathcal{Q}_{j,r}|$, it follows that

$$\frac{1}{C}\left(\frac{L}{r2^j}\right)^{d-1} \leq N_{j,r} \leq C\left(\frac{L}{r2^j}\right)^{d-1}.$$

We say that a set $D \in \mathcal{Q}_{j,r}$ is *bad* if $\mathcal{B}^\partial_{L,r} \cap D \neq \emptyset$. As we want to make use of independence, we partition $\mathcal{Q}_{j,r}$ into disjoint sets $\mathcal{Q}_{j,r}^{(1)}, \ldots, \mathcal{Q}_{j,r}^{(R)}$, such that for each $1 \leq m \leq R$ we have

- $\mathrm{d}(D, D') > 4 \max_{x \in \Lambda_j} h_{L,r}(x)$ for all $D \neq D' \in \mathcal{Q}_{j,r}^{(m)}$,
- $N_{j,r}^{(m)} = \left|\mathcal{Q}_{j,r}^{(m)}\right| \geq \frac{N_{j,r}}{2R}$.

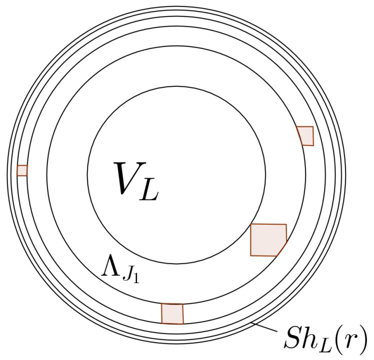

Figure 3: The layers Λ_j, $0 \le j \le J_1$, with $\Lambda_0 = \mathrm{Sh}_L(2r)$. Subsets $D_{\mathbf{k}}^{(j)} \subset \Lambda_j$ containing bad points are shaded.

Notice that $R \in \mathbb{N}$ can be chosen to depend on the dimension only. Then the events $\{D$ is bad$\}$, $D \in \mathcal{Q}_{j,r}^{(m)}$, are independent. Further, if $L_1 \le L \le L_1(\log L_1)^2$, it follows that under $\mathbf{C1}(\delta, L_1)$,

$$\mathbb{P}\left(D \text{ is bad}\right) \le C(r2^j)^{2d} \exp\left(-\left(\log(r2^j/20)\right)^2\right) \le \exp\left(-\left(\log r + j\right)^{5/3}\right) = p_{j,r},$$

for all $r \ge r_0$ and $j \in \mathbb{N}$, if r_0 is big enough. Let $Y_{j,r}$ and $Y_{j,r}^{(m)}$ be the number of bad sets in $\mathcal{Q}_{j,r}$ and $\mathcal{Q}_{j,r}^{(m)}$, respectively. For $r \ge 5$, we have $p_{j,r} \le (\log r + j)^{-3/2} \le 1/2$. A standard large deviation estimate for Bernoulli random variables yields

$$\mathbb{P}\left(Y_{j,r}^{(m)} \ge (\log r + j)^{-3/2} N_{j,r}^{(m)}\right) \le \exp\left(-N_{j,r}^{(m)}\ I\left((\log r + j)^{-3/2} \mid p_{j,r}\right)\right),$$

with $I(x \mid p) = x\log(x/p) + (1-x)\log((1-x)/(1-p))$. By enlarging r_0 if necessary, we get $I\left((\log r + j)^{-3/2} \mid p_{j,r}\right) \ge 2R(\log r + j)^{1/7}$ for $r \ge r_0$, whence

$$\begin{aligned}
&\mathbb{P}\left(Y_{j,r} \ge (\log r + j)^{-3/2} N_{j,r}\right) \\
&\le\ R \max_{m=1,\dots,R} \mathbb{P}\left(Y_{j,r}^{(m)} \ge (\log r + j)^{-3/2} N_{j,r}^{(m)}\right) \le R\exp\left(-(\log r + j)^{1/7} N_{j,r}\right) \\
&\le\ R\exp\left(-\frac{1}{C}\ (\log r + j)^{1/7}\left(\frac{L}{r2^j}\right)^{d-1}\right) \le \exp\left(-(\log r + j)^{1/7}(\log L)^{29}\right),
\end{aligned}$$

for $r_0 \le r < r_L$, $0 \le j \le J_1(L)$ and L large enough. In particular,

$$\sum_{0 \le j \le J_1(L)} \mathbb{P}\left(Y_{j,r} \ge (\log r + j)^{-3/2} N_{j,r}\right) \le \exp\left(-(\log L)^{28}\right).$$

Therefore, setting

$$\mathrm{BdBad}_{L,r} = \bigcup_{0 \le j \le J_1(L)} \left\{Y_{j,r} \ge (\log r + j)^{-3/2} N_{j,r}\right\},$$

we have proved the following

Lemma 2.2. *There exists a constant $r_0 > 0$ such that if $r \ge r_0$ and L_1 is large enough, then $\mathbf{C1}(\delta, L_1)$ implies that for L with $L_1 \le L \le L_1(\log L_1)^2$,*

$$\mathbb{P}\left(\mathrm{BdBad}_{L,r}\right) \le \exp\left(-(\log L)^{28}\right).$$

3 Some important estimates

In this section, we present various results on exit and hitting proabalities for both simple random walk and Brownian motion.

3.1 Hitting probabilities

The first two lemmata concern simple random walk.

Lemma 3.1. *Let $0 < \eta < 1$.*

(i) *There exists $C = C(\eta) > 0$ such that for all $x \in V_{\eta L}$, $y \in \partial V_L$,*

$$C^{-1}L^{-d+1} \leq \pi_L(x,y) \leq CL^{-d+1}.$$

(ii) *There exists $C = C(\eta) > 0$ such that for all $x, x' \in V_{\eta L}$, $y \in \partial V_L$,*

$$|\pi_L(x,y) - \pi_L(x',y)| \leq C|x-x'|L^{-d}.$$

(iii) *Let $0 < l < L$ and $x \in V_L$ with $l < |x| < L$. Then*

$$\mathrm{P}_x(\tau_L < T_{V_l}) = \frac{l^{-d+2} - |x|^{-d+2} + O(l^{-d+1})}{l^{-d+2} - L^{-d+2}}.$$

Proof: (i) $\pi_L(\cdot, y)$ is harmonic inside V_L. Applying a discrete Harnack inequality, as, for example, provided by Theorem 6.3.9 in the book of Lawler and Limic [24], we see that $C^{-1}\pi_L(0,y) \leq \pi_L(\cdot,y) \leq C\pi_L(0,y)$ on $V_{\eta L}$, for some $C = C(d,\eta)$. Part (i) then follows from Lemma 6.3.7 in the same book.
(ii) By the triangle inequality,

$$|\pi_L(x,y) - \pi_L(x',y)| \leq C|x-x'| \max_{u,v \in V_{\eta L}: |u-v| \leq 1} |\pi_L(u,y) - \pi_L(v,y)|.$$

For $u \in V_{\eta L}$, the function $\pi_L(u + \cdot, y)$ is harmonic inside $V_{(1-\eta)L}$. The claim now follows from [24] Theorem 6.3.8, (6.19), together with (i).
(iii) This is Proposition 1.5.10 of [23]. □

A good control over hitting probabilities is given by

Lemma 3.2. *Let $a \geq 1$ and $x, y \in \mathbb{Z}^d$ with $x \notin V_a(y)$. Then*

(i)

$$\mathrm{P}_x\left(T_{V_a(y)} < \infty\right) = \left(\frac{a}{|x-y|}\right)^{d-2} \left(1 + O(a^{-1})\right).$$

(ii) *There exists $C > 0$, independent of a, such that when $|x-y| > 7a$,*

$$\mathrm{P}_x\left(T_{V_a(y)} < \tau_L\right) \leq C\frac{a^{d-2}\max\{a, \mathrm{d}_L(y)\}\max\{1, \mathrm{d}_L(x)\}}{|x-y|^d}.$$

(iii) *There exists $C > 0$ such that for all $x \in V_L$, $y \in \partial V_L$,*

$$C^{-1} \frac{\mathrm{d}_L(x)}{|x-y|^d} \leq \pi_L(x,y) \leq C \frac{\max\{1, \mathrm{d}_L(x)\}}{|x-y|^d}.$$

This lemma will be proved in the appendix.

We need analogous results for Brownian motion in $\mathbb{R}^d$. Denote by $\pi_L^{\mathrm{BM}}(y, dz)$ the exit measure of d-dimensional Brownian motion from C_L, started at $y \in C_L$. By a small abuse of notation, we also write $\pi_L^{\mathrm{BM}}(y,z)$ for the (continuous version of the) density with respect to surface measure on C_L, which is given by the Poisson kernel

$$\pi_L^{\mathrm{BM}}(y,z) = \frac{1}{d\,\alpha(d)\,L} \frac{L^2 - |y|^2}{|y-z|^d}, \tag{8}$$

where $\alpha(d)$ is the volume of the unit ball. From this explicit form, we can directly read off the analogous statements of Lemma 3.1 (i), (ii) and Lemma 3.2 (iii), with V_L replaced by C_L. Let us now formulate and prove the analog of parts (i) and (ii) from the last lemma. Denote by $\mathrm{P}_x^{\mathrm{BM}}$ the law of standard d-dimensional Brownian motion, started at $x \in \mathbb{R}^d$. For the following statement, $T_{C_a(y)}$ and τ_{C_L} are defined in the obvious way in terms of Brownian motion.

Lemma 3.3. *Let $a > 0$ and $x, y \in \mathbb{R}^d$ with $x \notin C_a(y)$. Then*

(i)

$$\mathrm{P}_x^{\mathrm{BM}}\left(T_{C_a(y)} < \infty\right) = \left(\frac{a}{|x-y|}\right)^{d-2}.$$

(ii) *Assume $C_{2a}(y) \subset C_L$. There exists $K > 0$ such that*

$$\mathrm{P}_x^{\mathrm{BM}}\left(T_{C_a(y)} < \tau_{C_L}\right) \leq K \frac{a^{d-2}\,\mathrm{d}_L(y)\,\mathrm{d}_L(x)}{|x-y|^d}.$$

Proof of Lemma 3.3: (i) See for example the book of Durrett [12], (1.12).
(ii) Recall that the Green's function of Brownian motion for C_L is given by

$$g^{\mathrm{BM}}(x,y) = A_d \left(\left(\frac{1}{|x-y|}\right)^{d-2} - \left(\frac{L}{|x|\,|x^\star - y|}\right)^{d-2} \right),$$

where A_d is an explicit constant, and for $x \neq 0$, $x^\star = (L^2/|x|^2)x$ is the inversion of x with respect to C_L. Now, for $a > 0$ and $x \notin C_a(y)$, we have

$$\int_{C_{a/2}(y)} g^{\mathrm{BM}}(x,z)\mathrm{d}z \geq \mathrm{P}_x^{\mathrm{BM}}\left(T_{C_a(y)} < \tau_{C_L}\right) \inf_{v \in \partial C_a(y)} \int_{C_{a/2}(y)} g^{\mathrm{BM}}(v,z)\mathrm{d}z.$$

By Proposition 1 of [10], the infimum on the right-hand side can be bounded from below by $c\,a^2$. Using the second upper bound on $g^{\mathrm{BM}}(x,z)$ from the same proposition, we get

$$\mathrm{P}_x^{\mathrm{BM}}\left(T_{C_a(y)} < \tau_{C_L}\right) \leq c^{-1}a^{-2} \int\limits_{C_{a/2}(y)} g^{\mathrm{BM}}(x,z)\mathrm{d}z \leq K\, \frac{a^{d-2}\,\mathrm{d}_L(y)\,\mathrm{d}_L(x)}{|x-y|^d}.$$

□

Remark 3.1. Lemma 3.2 (ii) can be proved in the same way if $|x|, |y| \leq cL$ for some $c < 1$, for example by using Proposition 8.4.1 of [24], which is based on a coupling argument. Since we need an estimate including the case when x or y are near the boundary, we give a self-contained proof in the appendix.

Probabilities of the above type will often be estimated by the following

Lemma 3.4. *Let $a > 0$, $l, m \geq 1$ and $x \in \mathbb{Z}^d$. Set $R_l = V_l \backslash V_{l-1}$, $\alpha = \max\{||x| - l|\,, a\}$. Then for some constant $C = C(m) > 0$*

$$\sum_{y\in R_l} \frac{1}{(a+|x-y|)^m} \leq C \begin{cases} l^{d-(m+1)} & \text{for } 1 \leq m < d-1 \\ \max\{\log(l/\alpha), 1\} & \text{for } m = d-1 \\ \alpha^{d-(m+1)} & \text{for } m \geq d \end{cases}.$$

Proof: If $\alpha > l$, then the left-hand side is bounded by

$$Cl^{d-1}\alpha^{-m} \leq C \max\left\{\alpha^{d-(m+1)}, l^{d-(m+1)}\right\}.$$

If $\alpha \leq l$, we set $A_k = \{y \in R_l : |x-y| \in [(k-1)\alpha, k\alpha)\}$. Then, for all $k \geq 1$,

$$\max_{y\in A_k} \frac{1}{(a+|x-y|)^m} \leq 2^m k^{-m}\alpha^{-m}.$$

Since for $k\alpha \leq l/10$ we have $|A_k| \leq C\alpha(k\alpha)^{d-2}$, the claim then follows from

$$\begin{aligned} \sum_{y\in R_l} \frac{1}{(a+|x-y|)^m} &\leq C\left(\sum_{1\leq k\leq \lfloor l/(10\alpha)\rfloor} \frac{\alpha(k\alpha)^{d-2}}{(k\alpha)^m}\right) + Cl^{d-1}l^{-m} \\ &\leq C\alpha^{d-(m+1)} \sum_{1\leq k\leq \lfloor l/(10\alpha)\rfloor} k^{d-(m+2)} + Cl^{d-(m+1)}. \end{aligned}$$

□

3.2 Smoothed exit measures

We will compare exit laws of simple random walk with exit laws of Brownian motion. Given a field of positive real numbers $\psi = (m_x)_{x\in\mathbb{R}^d}$, we define the smoothed exit law from V_L of simple random walk as

$$\phi_{L,\psi}(x,z) = \pi_L\hat{\pi}_\psi(x,z) = \sum_{y\in\partial V_L} \pi_L(x,y)\frac{1}{m_y}\int_{\mathbb{R}_+} \varphi\left(\frac{t}{m_y}\right)\pi_{V_t(y)}(y,z)\mathrm{d}t.$$

Denoting by $\pi^{\mathrm{BM}}_{C_t(x)}(x,dz)$ the exit measure of d-dimensional Brownian motion from $C_t(x)$, started at x, we let analogous to (1),

$$\hat{\pi}^{\mathrm{BM}}_{\psi}(x,dz) = \frac{1}{m_x}\int_{\mathbb{R}_+}\varphi\left(\frac{t}{m_x}\right)\pi^{\mathrm{BM}}_{C_t(x)}(x,dz)\mathrm{d}t.$$

Then define the smoothed Brownian exit measure from C_L as

$$\phi^{\mathrm{BM}}_{L,\psi}(x,dz) = \pi^{\mathrm{BM}}_L\hat{\pi}^{\mathrm{BM}}_{\psi}(x,dz) = \int_{\partial C_L}\pi^{\mathrm{BM}}_L(x,dy)\frac{1}{m_y}\int_{\mathbb{R}_+}\varphi\left(\frac{t}{m_y}\right)\pi^{\mathrm{BM}}_{C_t(y)}(y,dz)\mathrm{d}t.$$

By $\phi^{\mathrm{BM}}_{L,\psi}(x,z)$ we denote the density of $\phi^{\mathrm{BM}}_{L,\psi}(x,dz)$ with respect to d-dimensional Lebesgue measure.

Lemma 3.5. *Let $\psi \in \mathcal{M}_L$. There exists a constant $C > 0$ such that*

(i)
$$\sup_{x\in V_L}\sup_{z\in\mathbb{Z}^d}\left|\left(\phi_{L,\psi} - \phi^{\mathrm{BM}}_{L,\psi}\right)(x,z)\right| \leq CL^{-(d+1/4)}.$$

(ii)
$$\sup_{z\in\mathbb{R}^d}\left|\left|D^i\phi^{\mathrm{BM}}_{L,\psi}(\cdot,z)\right|\right|_{C_L} \leq CL^{-(d+i)}, \quad i = 0,1,2,3.$$

(iii)
$$\sup_{x,x'\in V_L\cup\partial V_L}\sup_{z\in\mathbb{Z}^d}|\phi_{L,\psi}(x,z) - \phi_{L,\psi}(x',z)| \leq C\left(L^{-(d+1/4)} + |x-x'|L^{-(d+1)}\right).$$

For the proof, we refer to the appendix. The next proposition will be applied at the end of the proof of Lemma 5.2. At this point, the invariance condition **A1** comes into play. We give a general formulation in terms of a signed measure ν. Let us introduce the following notation. For $x = (x_1,\ldots,x_d) \in \mathbb{Z}^d$, put

$$\begin{aligned} x^{(i)} &= (x_1,\ldots,x_{i-1},-x_i,x_{i+1},\ldots,x_d), \\ x^{\leftrightarrow(i,j)} &= (x_1,\ldots,x_{i-1},x_j,x_{i+1},\ldots,x_{j-1},x_i,x_{j+1},\ldots,x_d), \text{ if } i<j. \end{aligned}$$

Proposition 3.1. *Let $l > 0$. Consider a measure ν on V_l with total mass zero satisfying*

(i) $\nu(x) = \nu(x^{(i)})$ *for all x and all $i = 1,\ldots,d$.*

(ii) $\nu(x) = \nu(x^{\leftrightarrow(i,j)})$ *for all x and all $i,j = 1,\ldots,d$, $i<j$.*

Then there exists $C > 0$ such that for $y' \in V_L$ with $V_l(y') \subset V_L$ and all $z \in \mathbb{Z}^d$, $\psi \in \mathcal{M}_L$,

$$\left|\sum_{y\in V_l(y')}\nu(y-y')\phi_{L,\psi}(y,z)\right| \leq C||\nu||_1\left(L^{-(d+1/4)} + \left(\frac{l}{L}\right)^3 L^{-d}\right).$$

Proof: Since the proof is the same for all $y' \in V_L$ with $V_l(y') \subset V_L$, we can assume $y' = 0$. By Lemma 3.5 (i),

$$\left| \sum_y \nu(y)\phi_{L,\psi}(y,z) - \sum_y \nu(y)\phi_{L,\psi}^{\mathrm{BM}}(y,z) \right| \leq C||\nu||_1 L^{-(d+1/4)}.$$

Taylor's expansion gives

$$\begin{aligned}
&\sum_y \nu(y)\phi_{L,\psi}^{\mathrm{BM}}(y,z) \\
&= \sum_y \nu(y)\left[\phi_{L,\psi}^{\mathrm{BM}}(y,z) - \phi_{L,\psi}^{\mathrm{BM}}(0,z)\right] \\
&= \sum_y \nu(y)\nabla_x\phi_{L,\psi}^{\mathrm{BM}}(0,z)\cdot y + \frac{1}{2}\sum_y \nu(y)y \cdot H_x\phi_{L,\psi}^{\mathrm{BM}}(0,z)y + R(\nu,0,z),
\end{aligned} \tag{9}$$

where $\nabla_x\phi_{L,\psi}^{\mathrm{BM}}$ is the gradient, $H_x\phi_{L,\psi}^{\mathrm{BM}}$ the Hessian of $\phi_{L,\psi}^{\mathrm{BM}}$ with respect to the first variable, and $R(\nu,0,z)$ is the remainder term, which can be bounded by Lemma 3.5 (ii), namely

$$|R(\nu,0,z)| \leq C||\nu||_1 \left(\frac{l}{L}\right)^3 L^{-d}.$$

Since ν satisfies property (i), the first summand on the right side of (9) vanishes. Due to the same reason, the second summand equals

$$\frac{1}{2}\sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}\phi_{L,\psi}^{\mathrm{BM}}(0,z)\sum_y \nu(y)(y_i)^2.$$

By property (ii), the sum over y does not depend on i, so a multiple of the Laplacian of $\phi_{L,\psi}^{\mathrm{BM}}$ remains. But for each $v \in \partial C_L$, $\pi_L^{\mathrm{BM}}(\cdot,v)$ is harmonic in C_L, thus also the Laplacian vanishes. This proves the proposition. $\square$

4 Green's functions for the ball

One principal task of our approach aims at developing good estimates on Green's functions for the ball of both coarse grained (goodified) RWRE as well as coarse grained simple random walk. The main result is Lemma 4.1. For the coarse grained simple random walk, the estimates on hitting probabilities of the last section together with Proposition 4.2 yield the right control.

On a certain class of environments, we need to modify the transition kernels in order to ensure that bad points are not visited too often by the coarse grained random walks. This modification will be described in Section 4.4.

4.1 A local central limit theorem

Let $m \geq 1$. Denote by $\hat{\pi}_m$ the coarse grained transition probabilities on $\mathbb{Z}^d$ belonging to the field $\psi = (m_x)_{x \in \mathbb{Z}^d}$, where $m_x = m$ is chosen constant in x. Notice that $\hat{\pi}_m$ is centered, and the covariances satisfy

$$\sum_{y \in \mathbb{Z}^d} (y_i - x_i)(y_j - x_j)\hat{\pi}_m(x, y) = \gamma_m \delta_i(j),$$

where for large m (recall the coarse graining scheme) $1/d < \gamma_m/m^2 < 4/d$.

Proposition 4.1 (Local central limit theorem). *Let $x, y \in \mathbb{Z}^d$. For $m \geq 1$ and all integers $n \geq 1$,*

$$\hat{\pi}_m^n(x, y) = \frac{1}{(2\pi\gamma_m n)^{d/2}} \exp\left(-\frac{|x-y|^2}{2\gamma_m n}\right) + O\left(m^{-d} n^{-(d+2)/2}\right).$$

For the corresponding Green's function $\hat{g}_{m,\mathbb{Z}^d}(x, y) = \sum_{n=0}^{\infty} \hat{\pi}_m^n(x, y)$ we obtain

Proposition 4.2. *Let $x, y \in \mathbb{Z}^d$. There exists $m_0 > 0$ such that if $m \geq m_0$, then*

(i) *For $|x - y| < 3m$,*

$$\hat{g}_{m,\mathbb{Z}^d}(x, y) = \delta_x(y) + O(m^{-d}).$$

(ii) *For $|x - y| \geq 3m$, there exists a constant $c(d) > 0$ such that*

$$\hat{g}_{m,\mathbb{Z}^d}(x, y) = \frac{c(d)}{\gamma_m |x-y|^{d-2}} + O\left(\frac{1}{|x-y|^d}\left(\log \frac{|x-y|}{m}\right)^d\right).$$

Here, the constants in the O-notation are independent of m and $|x - y|$.

In our applications, m will be a function of L. Although these results look rather standard, we cannot directly refer to the literature because we have to keep track of the m-dependency. We give a proof of both statements in the appendix.

We will use the last proposition to estimate the Green's function for the ball V_L, $\hat{g}_m(x, y) = \sum_{n=0}^{\infty} (1_{V_L}\hat{\pi}_m)^n (x, y)$. Clearly, $\hat{g}_m$ is bounded from above by $\hat{g}_{m,\mathbb{Z}^d}$, and more precisely, the strong Markov property shows

$$\hat{g}_m(x, y) = \mathrm{E}_{x,\hat{\pi}_m}\left[\sum_{k=0}^{\tau_L - 1} 1_{\{X_k = y\}}\right] = \hat{g}_{m,\mathbb{Z}^d}(x, y) - \mathrm{E}_{x,\hat{\pi}_m}\left[\hat{g}_{m,\mathbb{Z}^d}(X_{\tau_L}, y)\right]. \tag{10}$$

4.2 Estimates on coarse grained Green's functions

As we will show, the perturbation expansion enables us to control the goodified Green's function $\hat{G}^g_{L,r}$ essentially in terms of $\hat{g}_{L,r}$. The boundary region $\mathrm{Sh}_L(r)$ turns out to be problematic, since even for good x, we cannot estimate the variational distance between the transition kernels by δ. We therefore work in this (and only in this) section with slightly modified transition kernels $\tilde{\Pi}_{L,r}$, $\tilde{\pi}_{L,r}$, $\tilde{\Pi}^g_{L,r}$ in the enlarged ball V_{L+r}, taking the exit measure in $\mathrm{Sh}_L(r)$ from uncut balls $V_t(x) \subset V_{L+r}$, $t \in [h_{L,r}(x), 2h_{L,r}(x)]$. More precisely, setting $h_{L,r}(x) = (1/20)r$ for $x \notin C_L$, we let $\tilde{\pi}_{L,r}$ be the coarse grained simple random walk kernel under $\tilde{\psi} = (h_{L,r}(x))_{x \in V_{L+r}}$, that is

$$\tilde{\pi}_{L,r}(x,\cdot) = \frac{1}{h_{L,r}(x)} \int_{\mathbb{R}_+} \varphi\left(\frac{t}{h_{L,r}(x)}\right) \pi_{V_t(x) \cap V_{L+r}}(x,\cdot)\mathrm{d}t.$$

For the corresponding RWRE kernel, we forget about the environment on $V_{L+r} \backslash V_L$ and set

$$\tilde{\Pi}_{L,r}(x,\cdot) = \begin{cases} \frac{1}{h_{L,r}(x)} \int_{\mathbb{R}_+} \varphi\left(\frac{t}{h_{L,r}(x)}\right) \Pi_{V_t(x)}(x,\cdot)\mathrm{d}t & \text{for } x \in V_L \\ \tilde{\pi}_{L,r}(x,\cdot) & \text{for } x \in V_{L+r} \backslash V_L \end{cases}.$$

For all good $x \in V_L$ we now have $||(\tilde{\Pi}_{L,r} - \tilde{\pi}_{L,r})(x,\cdot)||_1 \leq \delta$, while for $x \in V_{L+r} \backslash V_L$, the difference vanishes anyway. The goodified version of $\tilde{\Pi}_{L,r}$ is then obtained in an analogous way to (7),

$$\tilde{\Pi}^g_{L,r}(x,\cdot) = \begin{cases} \tilde{\Pi}_{L,r}(x,\cdot) & \text{for } x \notin \mathcal{B}^\star_{L,r} \\ \tilde{\pi}_{L,r}(x,\cdot) & \text{for } x \in \mathcal{B}^\star_{L,r} \end{cases}.$$

We write $\tilde{G}_{L,r}, \tilde{g}_{L,r}$ and $\tilde{G}^g_{L,r}$ for the corresponding Green's functions on V_{L+r}. Note that

$$\hat{G}_{L,r} \leq \tilde{G}_{L,r}, \quad \hat{g}_{L,r} \leq \tilde{g}_{L,r}, \quad \hat{G}^g_{L,r} \leq \tilde{G}^g_{L,r} \quad \text{pointwise on } V_{L+r} \times (V_{L+r} \backslash \partial V_L). \tag{11}$$

Since we do not have exact expressions for $\tilde{g}_{L,r}$ or $\tilde{G}_{L,r}$, we will construct a (deterministic) kernel $\Gamma_{L,r}$ that bounds the Green's functions from above. For $x \in V_{L+r}$, set

$$\tilde{\mathrm{d}}(x) = \max\left(\frac{\mathrm{d}_{L+r}(x)}{2}, 3r\right), \quad a(x) = \min\left(\tilde{\mathrm{d}}(x), s_L\right).$$

Further, let

$$\Gamma^{(1)}_{L,r}(x,y) = \frac{\tilde{d}(x)\tilde{d}(y)}{a(y)^2(a(y) + |x-y|)^d}, \quad \Gamma^{(2)}_{L,r}(x,y) = \frac{1}{a(y)^2(a(y) + |x-y|)^{d-2}}.$$

The kernel $\Gamma_{L,r}$ is defined as the pointwise minimum

$$\Gamma_{L,r} = \min\left\{\Gamma^{(1)}_{L,r}, \Gamma^{(2)}_{L,r}\right\}. \tag{12}$$

We cannot derive pointwise estimates on the Green's functions in terms of $\Gamma_{L,r}$, but we can use this kernel to obtain upper bounds on neighborhoods $U(x) = V_{a(x)}(x) \cap V_{L+r}$.

Call a function $F : V_{L+r} \times V_{L+r} \to \mathbb{R}_+$ a *positive kernel.* Given two positive kernels F and G, we write $F \preceq G$ if for all $x, y \in V_{L+r}$,

$$F(x, U(y)) \leq G(x, U(y)),$$

where $F(x, U)$ stands for $\sum_{y \in U \cap \mathbb{Z}^d} F(x, y)$. Further, we write $F \asymp 1$, if there is a constant $C > 0$ such that for all $x, y \in V_{L+r}$,

$$\frac{1}{C} F(x, y) \leq F(\cdot, \cdot) \leq CF(x, y) \quad \text{on } U(x) \times U(y).$$

We adopt this notation to positive functions of one argument: For $f : V_{L+r} \to \mathbb{R}_+$, $f \asymp 1$ means that for some $C > 0$, $C^{-1} f(x) \leq f(\cdot) \leq Cf(x)$ on any $U(x) \subset V_{L+r}$. Finally, given $0 < \eta < 1$, we say that a positive kernel A on V_{L+r} is *η-smoothing*, if for all $x \in V_{L+r}$, $A(x, U(x)) \leq \eta$, and $A(x, y) = 0$ whenever $y \notin U(x)$.

Now we are in the position to formulate our main statement of this section. Recall our convention concerning constants: They only depend on the dimension unless stated otherwise.

Lemma 4.1.

(i) *There exists a constant $C_1 > 0$ such that*

$$\hat{g}_{L,r} \preceq C_1 \Gamma_{L,r} \quad \textit{and} \quad \tilde{g}_{L,r} \preceq C_1 \Gamma_{L,r}.$$

(ii) *There exists a constant $C > 0$ such that for small $\delta > 0$,*

$$\hat{G}^g_{L,r} \preceq C \Gamma_{L,r} \quad \textit{and} \quad \tilde{G}^g_{L,r} \preceq C \Gamma_{L,r}.$$

Remark 4.1. Thanks to (11), it suffices to prove the bounds for $\tilde{g}_{L,r}$ and $\tilde{G}^g_{L,r}$. For later use, we keep track of the constant in part (i) of the lemma.

We first prove part (i), which is a straightforward consequence of the estimates on hitting probabilities in Section 3 and the next lemma.

Lemma 4.2. *There exists $C > 0$ such that for all $x \in V_{L+r}$ and $y \in V_L$ with $\mathrm{d}_L(y) \geq 4s_L$,*

$$\tilde{g}_{L,r}(x, y) \leq C \begin{cases} \frac{1}{s_L^2 \max\{|x-y|, s_L\}^{d-2}}, & y \neq x \\ 1 & , \quad y = x \end{cases}.$$

Proof: If $x = y$, then the claim follows from transience of simple random walk. Now assume $x \neq y$, and always $\mathrm{d}_L(y) \geq 4s_L$. Consider first the case $|x - y| \leq s_L$. Let $\hat{g}_m$ be defined as in the beginning of Section 4.1. Recall our coarse graining scheme. With $m = s_L/20$ we have

$$\tilde{g}(x, y) \leq \hat{g}_m(x, y) + \sup_{v \in \mathrm{Sh}_L(2s_L)} \mathrm{P}_v \left(T_{V_{s_L}(y)} < \tau_{V_{L+r}} \right) \sup_{\substack{w: w \neq y, \\ |w-y| \leq s_L}} \tilde{g}(w, y).$$

Since

$$\sup_{v \in \mathrm{Sh}_L(2s_L)} \mathrm{P}_v\left(T_{V_{s_L}(y)} < \tau_{V_{L+r}}\right) < 1$$

uniformly in L, it follows from Proposition 4.2 that

$$\tilde{g}(x,y) \leq C \sup_{\substack{w: w \neq y, \\ |w-y| \leq s_L}} \hat{g}_m(w,y) \leq C \sup_{\substack{w: w \neq y, \\ |w-y| \leq s_L}} \hat{g}_{m,\mathbb{Z}^d}(w,y) \leq \frac{C}{s_L^d}.$$

If $|x-y| > s_L$ we use Lemma 3.2 (i) and the first case to get

$$\tilde{g}(x,y) \leq \mathrm{P}_x\left(T_{V_{s_L}(y)} < \infty\right) \sup_{\substack{w: w \neq y, \\ |w-y| \leq s_L}} \tilde{g}(w,y) \leq \frac{C}{s_L^2 |x-y|^{d-2}}.$$

□

Proof of Lemma 4.1 (i): It suffices to prove the bound for $\tilde{g}$. First we show that there exists a constant $C > 0$ such that for all $y \in V_{L+r}$,

$$\sup_{x \in V_{L+r}} \tilde{g}\left(x, U(y)\right) \leq C. \tag{13}$$

At first let $\mathrm{d}_{L+r}(y) \leq 6r$. Then $U(y) \subset \mathrm{Sh}_{L+r}(10r)$. We claim that

$$\sup_{x \in V_{L+r}} \tilde{g}\left(x, \mathrm{Sh}_{L+r}(10r)\right) \leq C. \tag{14}$$

for some $C > 0$. Indeed, if $z \in \mathrm{Sh}_{L+r}(10r)$, then $\tilde{\pi}(z, \cdot)$ is an (averaging) exit distribution from balls $V_l(z) \cap V_{L+r}$, where $l \geq r/20$. Using Lemma 3.1 (i), we find a constant $k_1 = k_1(d)$ such that starting at any $z \in \mathrm{Sh}_{L+r}(10r)$, V_{L+r} is left after k_1 steps with probability > 0, uniformly in z. This together with the strong Markov property implies (14). Next assume $6r < \mathrm{d}_{L+r}(y) \leq 6s_L$. Then $U(y) \subset S(y) = \mathrm{Sh}_{L+r}\left(\frac{1}{2}\mathrm{d}_{L+r}(y), 2\,\mathrm{d}_{L+r}(y)\right)$. We claim that

$$\sup_{x \in V_{L+r}} \tilde{g}\left(x, S(y)\right) \leq C. \tag{15}$$

For $z \in S(y)$, $\tilde{\pi}(z, \cdot)$ is an averaging exit distribution from balls $V_l(z)$, where $l \geq \mathrm{d}_{L+r}(y)/240$. By Lemma 3.1 (i), we find some small $0 < c < 1$ and a constant $k_2(c,d)$ such that after k_2 steps, the walk has probability > 0 to be in $\mathrm{Sh}_{L+r}\left(\frac{1-c}{2}\mathrm{d}_{L+r}(y)\right)$, uniformly in z and y. But starting in $\mathrm{Sh}_{L+r}\left(\frac{1-c}{2}\mathrm{d}_{L+r}(y)\right)$, Lemma 3.1 (iii) shows that with probability > 0, the ball V_{L+r} is left before $S(y)$ is visited again. Therefore (15) and hence (13) hold in this case. At last, let $\mathrm{d}_{L+r}(y) > 6s_L$. Then $\mathrm{d}_L(w) \geq 4s_L$ for $w \in U(y)$. Estimating

$$\tilde{g}(x,w) \leq 1 + \sup_{v: v \neq w} \tilde{g}(v,w),$$

we get with part (i) that

$$\sup_{w \in U(y)} \tilde{g}(x,w) \leq 1 + \frac{C}{s_L^d}.$$

Summing over $w \in U(y)$, (13) follows. Finally, note that for any $x \in V_{L+r}$,

$$\tilde{g}(x, U(y)) \leq \mathrm{P}_x\left(T_{U(y)} \leq \tau_{V_{L+r}}\right) \sup_{w \in U(y)} \tilde{g}(w, U(y)).$$

Now $\tilde{g} \preceq C\Gamma$ follows from (13) and the hitting estimates of Lemma 3.2. □

Let us now explain our strategy for proving part (ii). By version (5) of the perturbation expansion, we can express $\tilde{G}^g_{L,r}$ in a series involving $\tilde{g}_{L,r}$ and differences of exit measures. The Green's function $\tilde{g}_{L,r}$ is already controlled by means of $\Gamma_{L,r}$. Looking at (5), we thus have to understand what happens if $\Gamma_{L,r}$ is concatenated with certain smoothing kernels. This will be the content of Proposition 4.3.

We start with collecting some important properties of $\Gamma_{L,r}$, which will be used throughout this text. Define for $j \in \mathbb{N}$

$$\mathcal{L}_j = \{y \in V_L : j \leq \mathrm{d}_L(y) < j+1\}, \quad \mathcal{E}_j = \{y \in V_{L+r} : \tilde{\mathrm{d}}(y) \leq 3jr\}.$$

Lemma 4.3 (Properties of $\Gamma_{L,r}$)**.**

(i) *Both* $\tilde{\mathrm{d}}$ *and* a *are Lipschitz with constant* $1/2$. *Moreover, for* $x, y \in V_{L+r}$,

$$a(y) + |x-y| \leq a(x) + \frac{3}{2}|x-y|.$$

(ii)

$$\Gamma_{L,r} \asymp 1.$$

(iii) *For* $0 \leq j \leq 2s_L$, $x \in V_{L+r}$,

$$\sum_{y \in \mathcal{L}_j} \left(\max\left\{1, \frac{\tilde{\mathrm{d}}(x)}{a(y)}\right\} \frac{1}{(a(y)+|x-y|)^d} \right) \leq C \frac{1}{j \vee r}.$$

(iv) *For* $1 \leq j \leq \frac{1}{3r}s_L$,

$$\sup_{x \in V_{L+r}} \Gamma_{L,r}(x, \mathcal{E}_j) \leq C \log(j+1),$$

and for $0 \leq \alpha < 3$,

$$\sup_{x \in V_{L+r}} \Gamma_{L,r}\left(x, \mathrm{Sh}_L\left(s_L, L/(\log L)^\alpha\right)\right) \leq C(\log\log L)(\log L)^{6-2\alpha}.$$

(v) *For* $x \in V_{L+r}$, *in the case of constant* r,

$$\Gamma_{L,r}(x, V_L) \leq C \max\left\{\frac{\tilde{\mathrm{d}}(x)}{L}(\log L)^6,\ \left(\frac{\tilde{\mathrm{d}}(x)}{r} \wedge \log L\right)\right\}.$$

In the case $r = r_L$,

$$\Gamma_{L,r_L}(x, V_L) \leq C \max\left\{\frac{\tilde{\mathrm{d}}(x)}{L}(\log L)^6,\ \left(\frac{\tilde{\mathrm{d}}(x)}{r_L} \wedge \log\log L\right)\right\}.$$

Proof: (i) The second statement is a direct consequence of the Lipschitz property, which in turn follows immediately from the definitions of $\tilde{\mathrm{d}}$ and a.
(ii) As for $y' \in U(y)$, $\frac{1}{2}a(y) \le a(y') \le \frac{3}{2}a(y)$ and similarly with a replaced by $\tilde{\mathrm{d}}$, it suffices to show that for $x' \in U(x)$, $y' \in U(y)$,

$$\frac{1}{C}\left(a(y) + |x-y|\right) \le a(y') + |x'-y'| \le C\left(a(y) + |x-y|\right). \tag{16}$$

First consider the case $|x-y| \ge 4\max\{a(x), a(y)\}$. Then

$$a(y) + |x-y| \le 2a(y') + 2\left(|x-y| - a(x) - a(y)\right) \le 2\left(a(y') + |x'-y'|\right).$$

If $|x-y| \le 4a(y)$ then

$$a(y) + |x-y| \le 5a(y) \le 5a(y) + |x'-y'| \le 10\left(a(y') + |x'-y'|\right),$$

while for $|x-y| \le 4a(x)$, using part (i) in the first inequality,

$$a(y) + |x-y| \le a(x) + \frac{3}{2}|x-y| \le 7a(x) \le 14\left(a(y') + |x'-y'|\right).$$

This proves the first inequality in (16). The second one follows from

$$a(y') + |x'-y'| \le \frac{5}{2}a(y) + a(x) + |x-y| \le \frac{7}{2}\left(a(y) + |x-y|\right).$$

(iii) If $j \le 2s_L$ and $y \in \mathcal{L}_j$, then $a(y)$ is of order $j \vee r$. By Lemma 3.4 we have

$$\sum_{y \in \mathcal{L}_j} \frac{1}{(j \vee r + |x-y|)^d} \le C \min\left\{\frac{1}{j \vee r}, \frac{1}{|\mathrm{d}_{L+r}(x) - (j+r)|}\right\}.$$

It remains to show that

$$\max\left\{1, \frac{\tilde{\mathrm{d}}(x)}{j \vee r}\right\} \min\left\{\frac{1}{j \vee r}, \frac{1}{|\mathrm{d}_{L+r}(x) - (j+r)|}\right\} \le C\frac{1}{j \vee r}. \tag{17}$$

If $\tilde{\mathrm{d}}(x) \le (j \vee 3r)$, this is clear. If $\tilde{\mathrm{d}}(x) > (j \vee 3r)$, (17) follows from $|\,\mathrm{d}_{L+r}(x) - (j+r)| \ge \tilde{\mathrm{d}}(x)/2$.
(iv) If $\tilde{\mathrm{d}}(y) \le 3jr$, then $\mathrm{d}_L(y) \le 6jr$. Estimating Γ by $\Gamma^{(1)}$, we get

$$\Gamma(x, \mathcal{E}_j) \le C \sum_{i=0}^{6jr} \sum_{y \in \mathcal{L}_i} \frac{\tilde{\mathrm{d}}(x)}{a(y)} \frac{1}{(a(y) + |x-y|)^d}.$$

Now the first assertion of (iv) follows from (iii). The second is proved similarly, so we omit the details.
(v) Set $B = \{y \in V_L : \tilde{\mathrm{d}}(y) \le s_L \vee 2\,\tilde{\mathrm{d}}(x)\}$. For $y \in V_L \backslash B$, it holds that $a(y) = s_L$ and $|x-y| \ge \tilde{\mathrm{d}}(y)$. Therefore,

$$\Gamma\left(x, V_L \backslash B\right) \le \Gamma^{(1)}\left(x, V_L \backslash B\right) \le \frac{\tilde{\mathrm{d}}(x)}{s_L^2} \sum_{y \in V_{2L}} \frac{1}{(s_L + |y|)^{d-1}} \le C\frac{\tilde{\mathrm{d}}(x)}{L}(\log L)^6.$$

Furthermore,

$$\Gamma(x,B) \leq \sum_{i=0}^{2s_L} \sum_{y\in\mathcal{L}_i} \frac{\tilde{\mathrm{d}}(x)}{a(y)} \frac{1}{(a(y)+|x-y|)^d} + \frac{1}{s_L^2} \sum_{\substack{y\in V_L:\\ s_L\leq \tilde{\mathrm{d}}(y)\leq 2\,\tilde{\mathrm{d}}(x)}} \frac{1}{(s_L+|x-y|)^{d-2}}.$$

Lemma 3.4 bounds the second term by $C(\tilde{\mathrm{d}}(x)/L)(\log L)^6$. For the first term, we use twice part (iii) and once Lemma 3.4 to get

$$\sum_{i=0}^{2s_L} \sum_{y\in\mathcal{L}_i} \frac{\tilde{\mathrm{d}}(x)}{a(y)} \frac{1}{(a(y)+|x-y|)^d} \leq C\sum_{i=0}^{5r} \frac{1}{i\vee r} + C\min\left\{\tilde{\mathrm{d}}(x)\sum_{i=5r}^{2s_L}\frac{1}{i^2}, \sum_{i=5r}^{2s_L}\frac{1}{i}\right\}.$$

This proves (v). □

Proposition 4.3 (Concatenating). *Let F, G be positive kernels with $F \preceq G$.*

(i) *If A is η-smoothing and $G \asymp 1$, then for some constant $C = C(d,G) > 0$,*

$$FA \preceq C\eta G.$$

(ii) *If Φ is a positive function on V_{L+r} with $\Phi \asymp 1$, then for some $C = C(d,\Phi) > 0$,*

$$F\Phi \leq CG\Phi.$$

Proof: (i) As a is Lipschitz with constant $1/2$, we can choose $K = K(d)$ points y_k out of the set $M = \{y' \in V_{L+r} : U(y') \cap U(y) \neq \emptyset\}$ such that M is covered by the union of the $U(y_k), k = 1,\ldots,K$. Since $A(y', U(y)) \neq 0$ implies $y' \in M$, we then have

$$\begin{aligned} FA(x,U(y)) &= \sum_{y'\in M} F(x,y') \sum_{y''\in U(y)} A(y',y'') \leq \eta \sum_{k=1}^{K} F(x,U(y_k)) \\ &\leq \eta \sum_{k=1}^{K} G(x,U(y_k)). \end{aligned}$$

Using $G \asymp 1$, we get $G(x,U(y_k)) \leq C|U(y_k)|G(x,y)$. Clearly $|U(y_k)| \leq C|U(y)|$, so that

$$FA(x,U(y)) \leq CK\eta|U(y)|G(x,y).$$

A second application of $G \asymp 1$ yields the claim.
(ii) We can find a constant $K = K(d)$ and a covering of V_{L+r} by neighborhoods $U(y_k)$, $y_k \in V_{L+r}$, such that every $y \in V_{L+r}$ is contained in at most K many of the sets $U(y_k)$. Using $\Phi \asymp 1$, it follows that for $x \in V_{L+r}$,

$$\begin{aligned} F\Phi(x) = \sum_{y\in V_{L+r}} F(x,y)\Phi(y) &\leq C\sum_{k=1}^{\infty} F(x,U(y_k))\Phi(y_k) \leq C\sum_{k=1}^{\infty} G(x,U(y_k))\Phi(y_k) \\ &\leq C\sum_{k=1}^{\infty}\sum_{y\in U(y_k)} G(x,y)\Phi(y) \leq CK\sum_{y\in V_{L+r}} G(x,y)\Phi(y). \end{aligned}$$

□

In terms of our specific kernel $\Gamma_{L,r}$, we obtain

Proposition 4.4. *Let A be η-smoothing, and let F be a positive kernel satisfying $F \preceq \Gamma_{L,r}$.*

(i) *There exists a constant $C_2 > 0$ not depending on F and A such that*

$$FA \preceq C_2 \eta \Gamma_{L,r}.$$

(ii) *If additionally $A(x,y) = 0$ for $x \notin V_L$ and $A(x,U(x)) \leq (\log(a(x)/20))^{-9}$ for $x \in V_L \backslash \mathcal{E}_1$, then there exists a constant $C_3 > 0$ not depending on F and A such that for all $x, z \in V_{L+r}$,*

$$FA\Gamma_{L,r}(x,z) \leq C_3 \eta^{1/2} \Gamma_{L,r}(x,z).$$

Proof: (i) This is Proposition 4.3 (i) with $G = \Gamma$.
(ii) We set $B = V_L \backslash \mathcal{E}_1$ and split into

$$FA\Gamma = F1_{\mathcal{E}_1} A\Gamma + F1_B A\Gamma. \tag{18}$$

Let $x, z \in V_{L+r}$ be fixed, and consider first $F1_{\mathcal{E}_1} A\Gamma(x,z)$. Using $\Gamma \asymp 1$, $A\Gamma(y,z) \leq C\eta\Gamma(y,z)$. As $\Gamma(\cdot,z) \asymp 1$ and $F1_{\mathcal{E}_1} \preceq \Gamma 1_{\mathcal{E}_2}$, we get by Proposition 4.3 ii)

$$F1_{\mathcal{E}_1} A\Gamma(x,z) \leq C\eta\Gamma 1_{\mathcal{E}_2}\Gamma(x,z).$$

Setting $\mathcal{E}_2^1 = \{y \in \mathcal{E}_2 : |y-z| \geq |x-z|/2\}$, $\mathcal{E}_2^2 = \mathcal{E}_2 \backslash \mathcal{E}_2^1$, we split further into

$$\Gamma 1_{\mathcal{E}_2}\Gamma = \Gamma 1_{\mathcal{E}_2^1}\Gamma + \Gamma 1_{\mathcal{E}_2^2}\Gamma.$$

If $y \in \mathcal{E}_2^1$, then $\Gamma(y,z) \leq C\Gamma(x,z)$. By Lemma 4.3 (iv), $\Gamma(x,\mathcal{E}_2) \leq C$. Together we obtain

$$\Gamma 1_{\mathcal{E}_2^1}\Gamma(x,z) \leq C\Gamma(x,z).$$

If $y \in \mathcal{E}_2^2$, then $\Gamma(x,y) \leq C\frac{a(z)^2}{r^2}\Gamma(x,z)$ and $\Gamma^{(1)}(y,z) \leq C\frac{r^2}{a(z)^2}\Gamma^{(1)}(z,y)$, whence

$$\Gamma 1_{\mathcal{E}_2 \backslash \mathcal{E}_2^2}\Gamma(x,z) \leq C\Gamma(x,z)\Gamma^{(1)}(z,\mathcal{E}_2) \leq C\Gamma(x,z).$$

We therefore have shown that

$$F1_{\mathcal{E}_1} A\Gamma(x,z) \leq C\eta\Gamma(x,z).$$

To handle the second summand of (18), set $\sigma(y) = \min\left\{\eta, (\log a(y))^{-9}\right\}$, $y \in V_{L+r}$. Clearly, $1_B A\Gamma(y,z) \leq C\sigma(y)\Gamma(y,z)$ and $F1_B \preceq \Gamma 1_{V_L}$. Furthermore, $\sigma(\cdot)\Gamma(\cdot,z) \asymp 1$, so that by Proposition 4.3 ii)

$$F1_B A\Gamma(x,z) \leq C\Gamma 1_{V_L}\sigma\Gamma(x,z).$$

Consider $D^1 = \{y \in V_L : |y-z| \geq |x-z|/2\}$, $D^2 = V_L \backslash D^1$ and split into

$$\Gamma 1_{V_L} \sigma \Gamma = \Gamma 1_{D^1} \sigma \Gamma + \Gamma 1_{D^2} \sigma \Gamma.$$

If $y \in D^1$, then $\Gamma(y,z) \leq C \max\left\{1, \frac{\tilde{\mathrm{d}}(y)}{\tilde{\mathrm{d}}(x)}\right\} \Gamma(x,z)$, implying $\Gamma 1_{D^1} \sigma \Gamma(x,z) \leq C\eta^{1/2}\Gamma(x,z)$ if we prove

$$\sum_{y \in V_L} \max\left\{1, \frac{\tilde{\mathrm{d}}(y)}{\tilde{\mathrm{d}}(x)}\right\} \Gamma(x,y)\sigma(y) \leq C\eta^{1/2}. \tag{19}$$

To this end, we treat the summation over $S^1 = \{y \in V_L : \mathrm{d}_L(y) \leq 2s_L\}$ and $S^2 = V_L \backslash S_1$ separately. If $y \in S^2$, then $a(y) = s_L$. Estimating Γ by $\Gamma^{(1)}$ and $\tilde{\mathrm{d}}(y)$, $\tilde{\mathrm{d}}(x)$ simply by L, we get

$$\sum_{y \in S^2} \max\left\{1, \frac{\tilde{\mathrm{d}}(y)}{\tilde{\mathrm{d}}(x)}\right\} \Gamma(x,y)\sigma(y) \leq \frac{C}{(\log L)^3} \sum_{y \in V_{2L}} \frac{1}{(s_L + |y|)^d} \leq \frac{C \log\log L}{(\log L)^3}. \tag{20}$$

If $y \in S^1$, we estimate Γ again by $\Gamma^{(1)}$ and split the summation into the layers $\mathcal{L}_j$, $j = 0, \ldots, 2s_L$. On $\mathcal{L}_j$, $\sigma(y) \leq C \min\{\eta, (\log(j+1))^{-9}\}$. Thus, by Lemma 4.3 (iii),

$$\begin{aligned}
&\sum_{y \in S^1} \max\left\{1, \frac{\tilde{\mathrm{d}}(y)}{\tilde{\mathrm{d}}(x)}\right\} \Gamma(x,y)\sigma(y) \\
&\leq \; C \sum_{j=0}^{2s_L} \sum_{y \in \mathcal{L}_j} \max\left\{1, \frac{\tilde{\mathrm{d}}(x)}{a(y)}\right\} \frac{\min\{\eta, (\log(j+1))^{-9}\}}{(a(y) + |x-y|)^d} \\
&\leq \; C \sum_{j=0}^{2s_L} \frac{\min\{\eta, (\log(j+1))^{-9}\}}{j \vee r} \leq C\eta^{1/2}.
\end{aligned}$$

Together with (20), we have proved (19). It remains to bound the term $\Gamma 1_{D^2} \sigma \Gamma(x,z)$. But if $y \in D^2$, then

$$a(y) + |x-y| \geq a(y) + \frac{1}{2}|x-z| \geq a(z) - \frac{1}{2}|y-z| + \frac{1}{2}|x-z| \geq \frac{1}{4}\left(a(z) + |x-z|\right),$$

whence $\Gamma(x,y) \leq C\frac{a(z)^2}{a(y)^2} \max\left\{1, \frac{\tilde{\mathrm{d}}(y)}{\tilde{\mathrm{d}}(z)}\right\} \Gamma(x,z)$. Using Lemma 4.3 (i), we have

$$\frac{a(z)^2}{a(y)^2}\Gamma(y,z) \leq C\Gamma(z,y),$$

so that $\Gamma 1_{D^2} \sigma \Gamma(x,z) \leq C\eta^{1/2}\Gamma(x,z)$ follows again from (19). $\square$

Now we have collected all ingredients to finally prove part (ii) of our main Lemma 4.1.
Proof of Lemma 4.1 (ii): As already remarked, we only have to prove the statement involving $\tilde{G}^g$. The perturbation expansion (5) yields

$$\tilde{G}^g = \tilde{g} \sum_{m=0}^{\infty} (R\tilde{g})^m \sum_{k=0}^{\infty} \Delta^k,$$

where $\Delta = 1_{V_{L+r}}(\tilde{\Pi}^g - \tilde{\pi})$, $R = \sum_{k=1}^{\infty} \Delta^k \tilde{\pi}$. With the constants C_1 of Lemma 4.1 (i) and C_2, C_3 of Proposition 4.4 we choose

$$\delta \leq \frac{1}{16}\left(\frac{1}{C_2 \vee C_1^2 C_3^2}\right).$$

From Lemma 4.1 (i) and Proposition 4.4 (i) with $A = |\Delta|$, $\eta = \delta$ we then deduce that $\tilde{g}|\Delta| \preceq (C_1/2)\Gamma$, and, by iterating,

$$\sum_{k=1}^{\infty} \tilde{g}|\Delta|^{k-1} \preceq 2C_1\Gamma.$$

Furthermore, by part (ii) of Proposition 4.4 with $A = |\Delta\tilde{\pi}|$ and Lemma 4.1 (i),

$$\sum_{k=1}^{\infty} \tilde{g}|\Delta|^{k-1} \, |\Delta\tilde{\pi}| \, \tilde{g} \preceq (C_1/2)\Gamma.$$

Repeating this procedure shows that for $m \in \mathbb{N}$,

$$\tilde{g}(|R|\tilde{g})^m \preceq C_1 2^{-m}\Gamma.$$

Finally, by a further application of Proposition 4.4 (i),

$$\tilde{g}\sum_{m=0}^{\infty}(|R|\tilde{g})^m \sum_{k=0}^{\infty} |\Delta|^k \preceq 4C_1\Gamma.$$

This proves the lemma. □

4.3 Difference estimates

The results from the preceding section enable us to prove some difference estimates on the coarse grained Green's functions, which will be used in the part on mean sojourn times. The reader who is only interested in the exit measures may skip this section.

Lemma 4.4. *There exists a constant $C > 0$ such that*

(i)
$$\sup_{x,x' \in V_L : |x-x'| \leq s_L} \sum_{y \in V_L} |\hat{g}_{L,r}(x,y) - \hat{g}_{L,r}(x',y)| \leq C(\log\log L)(\log L)^3.$$

(ii) *For $\delta > 0$ small,*

$$\sup_{x,x' \in V_L : |x-x'| \leq s_L} \sum_{y \in V_L} \left|\hat{G}^g_{L,r_L}(x,y) - \hat{G}^g_{L,r_L}(x',y)\right| \leq C(\log\log L)(\log L)^3.$$

Proof: (i) Set $m = s_L/20$. Recall the definitions of $\hat{\pi}_m$ and $\hat{g}_m$ from Section 4.1. We write

$$\begin{aligned}&\sum_{y\in V_L} |\hat{g}(x,y) - \hat{g}(x',y)| \\ &\quad\leq \sum_{y\in V_L} |(\hat{g}-\hat{g}_m)(x,y)| + \sum_{y\in V_L} |\hat{g}_m(x,y) - \hat{g}_m(x',y)| + \sum_{y\in V_L} |(\hat{g}_m-\hat{g})(x',y)|.\end{aligned} \tag{21}$$

If $x \in V_L \setminus \mathrm{Sh}_L(2s_L)$, we have $\hat{\pi}(x,\cdot) = \hat{\pi}_m(x,\cdot)$. Clearly, $\sup_{x\in V_L} \hat{g}_m(x, \mathrm{Sh}_L(2s_L)) \leq C$. Thus, with $\Delta = 1_{V_L}(\hat{\pi}_m - \hat{\pi})$, expansion (3) and Lemma 4.3 yield (remember $\hat{g} \preceq C\Gamma$)

$$\begin{aligned}\sum_{y\in V_L} |(\hat{g}_m - \hat{g})(x,y)| &= \sum_{y\in V_L} |\hat{g}_m \Delta \hat{g}(x,y)| \\ &\leq 2\,\hat{g}_m(x, \mathrm{Sh}_L(2s_L)) \sup_{v\in \mathrm{Sh}_L(3s_L)} \hat{g}(v, V_L) \leq C(\log L)^3.\end{aligned}$$

It remains to handle the middle term of (21). By (10),

$$\begin{aligned}&\hat{g}_m(x,y) - \hat{g}_m(x',y) \\ &\quad= \hat{g}_{m,\mathbb{Z}^d}(x,y) - \hat{g}_{m,\mathbb{Z}^d}(x',y) + \mathrm{E}_{x',\hat{\pi}_m}\left[\hat{g}_{m,\mathbb{Z}^d}(X_{\tau_L}, y)\right] - \mathrm{E}_{x,\hat{\pi}_m}\left[\hat{g}_{m,\mathbb{Z}^d}(X_{\tau_L}, y)\right].\end{aligned}$$

Using Proposition 4.2, it follows that for $|x - x'| \leq s_L$,

$$\sum_{y\in V_L} \left|\hat{g}_{m,\mathbb{Z}^d}(x,y) - \hat{g}_{m,\mathbb{Z}^d}(x',y)\right| \leq C(\log L)^3.$$

At last, we claim that

$$\sum_{y\in V_L} \left|\mathrm{E}_{x',\hat{\pi}_m}\left[\hat{g}_{m,\mathbb{Z}^d}(X_{\tau_L}, y)\right] - \mathrm{E}_{x,\hat{\pi}_m}\left[\hat{g}_{m,\mathbb{Z}^d}(X_{\tau_L}, y)\right]\right| \leq C(\log\log L)(\log L)^3. \tag{22}$$

Since $|x - x'| \leq m$, we can define on the same probability space, whose probability measure we denote by $\mathbb{Q}$, a random walk $(Y_n)_{n\geq 0}$ starting at x and a random walk $(\tilde{Y}_n)_{n\geq 0}$ starting at x', both moving according to $\hat{\pi}_m$ on $\mathbb{Z}^d$, such that for all times n, $|Y_n - \tilde{Y}_n| \leq s_L$. However, with $\tau = \inf\{n \geq 0 : Y_n \notin V_L\}$, $\tilde{\tau}$ the same for $\tilde{Y}_n$, we cannot deduce that $|Y_\tau - \tilde{Y}_{\tilde{\tau}}| \leq s_L$, since it is possible that one of the walks, say Y_n, exits V_L and then moves far away from the exit point, while staying close to both V_L and the walk $\tilde{Y}_n$, which might still be inside V_L. In order to show that such an event has a small probability, we argue in a similar way to [24], Proposition 7.7.1. Define

$$\sigma(s_L) = \inf\{n \geq 0 : Y_n \in \mathrm{Sh}_L(s_L)\},$$

and analogously $\tilde{\sigma}(s_L)$. Let $\vartheta = \sigma(s_L) \wedge \tilde{\sigma}(s_L)$. Since $|Y_\vartheta - \tilde{Y}_\vartheta| \leq s_L$,

$$\sigma(2s_L) \vee \tilde{\sigma}(2s_L) \leq \vartheta.$$

For $k \geq 1$, we introduce the events

$$\begin{aligned} B_k &= \left\{ \left| Y_i - Y_{\sigma(2s_L)} \right| > ks_L \text{ for all } i = \sigma(2s_L), \dots, \tau \right\}, \\ \tilde{B}_k &= \left\{ \left| \tilde{Y}_i - \tilde{Y}_{\tilde{\sigma}(2s_L)} \right| > ks_L \text{ for all } i = \tilde{\sigma}(2s_L), \dots, \tilde{\tau} \right\}. \end{aligned}$$

By the strong Markov property and the gambler's ruin estimate of [24], p. 223 (7.26),

$$\mathbb{Q}\left(B_k \cup \tilde{B}_k \right) \leq C_1/k$$

for some $C_1 > 0$ independent of k. Applying the triangle inequality to

$$Y_\tau - \tilde{Y}_{\tilde{\tau}} = (Y_\tau - Y_\vartheta) + \left(Y_\vartheta - \tilde{Y}_\vartheta \right) + \left(\tilde{Y}_\vartheta - \tilde{Y}_{\tilde{\tau}} \right),$$

we deduce, for $k \geq 3$,

$$\mathbb{Q}\left(\left| Y_\tau - \tilde{Y}_{\tilde{\tau}} \right| \geq ks_L \right) \leq 2C_1/(k-1).$$

Since $|Y_\tau - \tilde{Y}_{\tilde{\tau}}| \leq 2(L + s_L) \leq 3L$, it follows that

$$\mathbb{E}_{\mathbb{Q}}\left[\left| Y_\tau - \tilde{Y}_{\tilde{\tau}} \right| \right] \leq \sum_{k=1}^{3L} \mathbb{Q}\left(\left| Y_\tau - \tilde{Y}_{\tilde{\tau}} \right| \geq k \right) \leq C(\log\log L)s_L.$$

Also, for v, w outside and y inside V_L,

$$\left| \frac{1}{|v-y|^{d-2}} - \frac{1}{|w-y|^{d-2}} \right| \leq C \frac{|v-w|}{(L+1-|y|)^{d-1}}.$$

By Proposition 4.2, (22) now follows from summing over $y \in V_L$.
(ii) Let $x, x' \in V_L$ with $|x - x'| \leq s_L$ and set $\Delta = 1_{V_L}(\hat{\Pi}^g - \hat{\pi})$. With $B = V_L \setminus \mathrm{Sh}_L(2r_L)$,

$$\hat{G}^g = \hat{g} 1_B \Delta \hat{G}^g + \hat{g} 1_{B^c} \Delta \hat{G}^g + \hat{g}.$$

Replacing successively $\hat{G}^g$ in the first summand on the right-hand side,

$$\hat{G}^g = \sum_{k=0}^{\infty} (\hat{g} 1_B \Delta)^k \hat{g} + \sum_{k=0}^{\infty} (\hat{g} 1_B \Delta)^k \hat{g} 1_{B^c} \Delta \hat{G}^g = F + F 1_{B^c} \Delta \hat{G}^g,$$

where we have set $F = \sum_{k=0}^{\infty} (\hat{g} 1_B \Delta)^k \hat{g}$. With $R = \sum_{k=1}^{\infty} (1_B \Delta)^k \hat{\pi}$, expansion (5) gives

$$F = \hat{g} \sum_{m=0}^{\infty} (R\hat{g})^m \sum_{k=0}^{\infty} (1_B \Delta)^k = \hat{g} \sum_{k=0}^{\infty} (1_B \Delta)^k + \hat{g} R F. \tag{23}$$

Following the proof of Lemma 4.1 (ii), one deduces $|F| \preceq C\Gamma$. By Lemma 4.3 (iv) and (v), we see that for large L, uniformly in $x \in V_L$,

$$|F 1_{B^c} \Delta \hat{G}^g(x, V_L)| \leq C\Gamma(x, \mathrm{Sh}_L(2r_L)) \sup_{v \in \mathrm{Sh}_L(3r_L)} \Gamma(v, V_L) \leq C \log\log L.$$

Therefore,

$$\sum_{y\in V_L}\left|\hat{G}^g(x,y)-\hat{G}^g(x',y)\right|\le C\log\log L+\sum_{y\in V_L}|F(x,y)-F(x',y)|\,.$$

Using (23) and twice part (i),

$$\begin{aligned}&\sum_{y\in V_L}|F(x,y)-F(x',y)|\\ &\quad\le\sum_{y\in V_L}\left|\hat{g}\sum_{k=0}^{\infty}(1_B\Delta)^k(x,y)-\hat{g}\sum_{k=0}^{\infty}(1_B\Delta)^k(x',y)\right|+\sum_{y\in V_L}|\hat{g}RF(x,y)-\hat{g}RF(x',y)|\,.\end{aligned}\tag{24}$$

The first expression on the right is estimated by

$$\sum_{y\in V_L}\left|\sum_{w\in V_L}(\hat{g}(x,w)-\hat{g}(x',w))\sum_{k=0}^{\infty}(1_B\Delta)^k(w,y)\right|\le C(\log\log L)(\log L)^3,$$

where we have used part (i) and the fact that $||1_B\Delta(w,\cdot)||_1\le\delta$. The second factor of (24) is again bounded by (i) and the fact that for $u\in V_L$,

$$\begin{aligned}\sum_{y\in V_L}|RF(u,y)| &= \sum_{y\in V_L}\left|\sum_{k=1}^{\infty}(1_B\Delta)^k\hat{\pi}F(u,y)\right|\\ &\le \sum_{k=0}^{\infty}||1_B\Delta(u,\cdot)||_1^k\sup_{v\in B}||1_B\Delta\hat{\pi}(v,\cdot)||_1\sup_{w\in V_L}\sum_{y\in V_L}|F(w,y)|\\ &\le C(\log L)^{-9+6}=C(\log L)^{-3}.\end{aligned}$$

Altogether, this proves part (ii). □

4.4 Modified transitions on environments bad on level 4

We shall now describe an environment-depending second version of the coarse graining scheme, which leads to modified transition kernels $\breve{\Pi}_{L,r}$, $\breve{\Pi}^g_{L,r}$, $\breve{\pi}_{L,r}$ on "really bad" environments.

Assume $\omega\in\mathrm{OneBad}_L$ is bad on level 4, with $\mathcal{B}_L(\omega)\subset V_{L/2}$. Then there exists $D=V_{4h_L(z)}(z)\in\mathcal{D}_L$ with $\mathcal{B}_L(\omega)\subset D$, $z\in V_{L/2}$. On D, $cr_L\le h_{L,r}(\cdot)\le Cr_L$. By Lemma 4.1 and the definition of $\Gamma_{L,r}$, it follows easily that we can find a constant $K_1\ge 2$, depending only on d, such that whenever $|x-y|\ge K_1h_{L,r}(y)$ for some $y\in\mathcal{B}_L$, we have

$$\hat{G}^g_{L,r}(x,\mathcal{B}_L)\le C\Gamma_{L,r}(x,D)\le\frac{1}{10}.\tag{25}$$

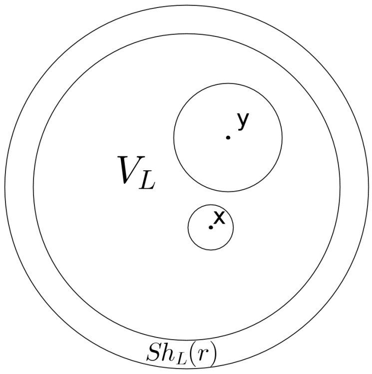

Figure 4: $\omega \in \text{OneBad}_L$ bad on level 4, with $\mathcal{B}_L \subset V_{L/2}$. The point x is “good”, so the coarse graining radii do not change at x. The point y is “bad”. Therefore, at y, the exit distribution is taken from the larger set $V_{t(y)}(y)$, where $t(y) = K_1 h_{L,r}(y)$.

On such ω, we let $t(x) = K_1 h_{L,r}(x)$ and define on V_L,

$$\breve{\Pi}_{L,r}(x,\cdot) = \begin{cases} \text{ex}_{V_{t(x)}(x)}\left(x,\cdot;\hat{\Pi}_{L,r}\right) & \text{for } x \in \mathcal{B}_L \\ \hat{\Pi}_{L,r}(x,\cdot) & \text{otherwise} \end{cases}.$$

By replacing $\hat{\Pi}$ by $\hat{\pi}$ on the right side, we define $\breve{\pi}_{L,r}(x,\cdot)$ in an analogous way. Note that $\breve{\pi}_{L,r}$ depends on the environment. We work again with a goodified version of $\breve{\Pi}_{L,r}$,

$$\breve{\Pi}^g_{L,r}(x,\cdot) = \begin{cases} \text{ex}_{V_{t(x)}(x)}\left(x,\cdot;\hat{\Pi}^g_{L,r}\right) & \text{for } x \in \mathcal{B}_L \\ \hat{\Pi}^g_{L,r}(x,\cdot) & \text{otherwise} \end{cases}.$$

For all other environments falling not into the above class, we change nothing and put $\breve{\Pi}_{L,r} = \hat{\Pi}_{L,r}$, $\breve{\Pi}^g_{L,r} = \hat{\Pi}^g_{L,r}$, $\breve{\pi}_{L,r} = \hat{\pi}_{L,r}$. This defines $\breve{\Pi}_{L,r}$, $\breve{\Pi}^g_{L,r}$ and $\breve{\pi}_{L,r}$ on all environments. We write $\breve{G}_{L,r}$, $\breve{G}^g_{L,r}$, $\breve{g}_{L,r}$ for the Green's functions corresponding to $\breve{\Pi}_{L,r}$, $\breve{\Pi}^g_{L,r}$ and $\breve{\pi}_{L,r}$.

Some properties of the new transition kernels

The following observations can be read off the definition and will be tacitly used below.

- On environments which are good or bad on level at most 3, the new kernels agree with the old ones, and so do their Green's functions, i.e. $\hat{G}_{L,r} = \breve{G}_{L,r}$ and $\hat{G}^g_{L,r} = \breve{G}^g_{L,r}$. On Good_L with the choice $r = r_L$, we have equality of all four Green's functions.

- If ω is not bad on level 4 with $\mathcal{B}_L \subset V_{L/2}$, then

$$1_{V_L}(\breve{\Pi}_{L,r} - \breve{\Pi}^g_{L,r}) = 1_{V_L}(\hat{\Pi}_{L,r} - \hat{\Pi}^g_{L,r}) = 1_{\mathcal{B}^\star_{L,r}}(\hat{\Pi} - \hat{\pi}).$$

  This will be used in Sections 5.2 and 6.

- In contrast to $\hat{\pi}_{L,r}$, the kernel $\breve{\pi}_{L,r}$ depends on the environment, too. However, $\breve{\Pi}_{L,r}$, $\breve{\Pi}^g_{L,r}$ and $\breve{\pi}_{L,r}$ do not change the exit measure from V_L, i.e. for example,

$$\mathrm{ex}_{V_L}\left(x,\cdot;\breve{\Pi}^g_{L,r}\right) = \mathrm{ex}_{V_L}\left(x,\cdot;\hat{\Pi}^g_{L,r}\right).$$

- The old transition kernels are finer in the sense that the (new) Green's functions $\breve{G}$, $\breve{G}^g$, $\breve{g}$ are pointwise bounded from above by $\hat{G}$, $\hat{G}^g$ and $\hat{g}$, respectively. In particular, we obtain with the same constants as in Lemma 4.1,

Lemma 4.5.

(i)

$$\breve{g}_{L,r} \preceq C_1 \Gamma_{L,r}.$$

(ii) *For $\delta > 0$ small,*

$$\breve{G}^g_{L,r} \preceq C\Gamma_{L,r}.$$

For the new goodified Green's function, we have

Corollary 4.1. *There exists a constant $C > 0$ such that for $\delta > 0$ small,*

(i) *On* OneBad_L*, if $\mathcal{B}_L \cap \mathrm{Sh}_L(r_L) = \emptyset$ or for general $\mathcal{B}_L$ in the case $r = r_L$,*

$$\sup_{x\in V_L} \breve{G}^g_{L,r}(x,\mathcal{B}_L) \le C.$$

On OneBad_L*, if $\mathcal{B}_L \not\subset V_{L/4}$, then, with $t = \mathrm{d}(\mathcal{B}_L, \partial V_L)$,*

$$\sup_{x\in V_{L/5}} \breve{G}^g_{L,r}(x,\mathcal{B}_L) \le C\left(\frac{s_L \wedge (t \vee r_L)}{L}\right)^{d-2}.$$

(ii) *On* $(\mathrm{BdBad}_{L,r})^c$, $\sup_{x\in V_{2L/3}} \breve{G}^g_{L,r}\left(x,\mathcal{B}^\partial_{L,r}\right) \le C(\log r)^{-1/2}$.

(iii) *For $\omega \in \mathrm{OneBad}_L$ bad on level at most 3 with $\mathcal{B}_L \cap \mathrm{Sh}_L(r_L) = \emptyset$, or for ω bad on level 4 with $\mathcal{B}_L \subset V_{L/2}$, putting $\Delta = 1_{V_L}(\breve{\Pi}_{L,r} - \breve{\Pi}^g_{L,r})$,*

$$\sup_{x\in V_L} \sum_{k=0}^{\infty} \left\| \left(\breve{G}^g_{L,r} 1_{\mathcal{B}_L} \Delta\right)^k (x,\cdot)\right\|_1 \le C.$$

Proof: (i) The set $\mathcal{B}_L$ is contained in a neighborhood $D \in \mathcal{D}_L$. As $\breve{G}^g \preceq C\Gamma$, we have

$$\breve{G}^g(x,\mathcal{B}_L) \le C\Gamma^{(2)}(x,D). \tag{26}$$

From this, the first statement of (i) follows. Now let x be inside $V_{L/5}$, and $\mathcal{B}_L \not\subset V_{L/4}$. If the midpoint z of D can be chosen to lie inside $V_L \setminus \mathrm{Sh}(r_L)$, $a(\cdot)/h_L(z)$ and $h_L(z)/a(\cdot)$

are bounded on D. Then, the second statement of (i) is again a consequence of (26). If $z \in \mathrm{Sh}(r_L)$, we have

$$\begin{aligned}\breve{G}^g(x, \mathcal{B}_L) \leq C\Gamma^{(1)}(x, D) &\leq C\sum_{j=0}^{2r_L} \sum_{y \in \mathcal{L}_j \cap D} \frac{L}{a(y)L^d} \\ &\leq CL^{-d+1} \sum_{j=0}^{2r_L} \frac{r_L^{d-1}}{j \vee r} \leq C(\log L)\left(\frac{r_L}{L}\right)^{d-1}.\end{aligned}$$

(ii) Recall the notation of Section 2.4. In order to bound $\sup_{x \in V_{2L/3}} \breve{G}^g(x, \mathcal{B}^\partial_{L,r})$, we look at the different bad sets $D_{j,r} \in \mathcal{Q}_{j,r}$ of layer Λ_j, $0 \leq j \leq J_1$. Estimating $\breve{G}^g$ by $\Gamma^{(1)}$, we have

$$\breve{G}^g(x, D_{j,r}) \leq C(r2^j)^{d-1}L^{-d+1}.$$

On $(\mathrm{BdBad}_{L,r})^c$, the number of bad sets in layer Λ_j is bounded by

$$C(\log r + j)^{-3/2}(L/(r2^j))^{d-1}.$$

Therefore,

$$\breve{G}^g\left(x, \mathcal{B}^\partial_{L,r} \cap \Lambda_j\right) \leq C(\log r + j)^{-3/2}.$$

Summing over $0 \leq j \leq J_1$, this shows

$$\breve{G}^g\left(x, \mathcal{B}^\partial_{L,r}\right) \leq C(\log r)^{-1/2}.$$

(iii) Assume $\omega \in \mathrm{Good}_L$ or ω is bad on level $i = 1, 2, 3$. Then $1_{\mathcal{B}_L}\Delta = 1_{\mathcal{B}_L}(\hat{\Pi} - \hat{\pi})$. Further, if $\mathcal{B}_L \cap \mathrm{Sh}_L(r_L) = \emptyset$, we have $||\breve{G}^g 1_{\mathcal{B}_L}\Delta(x, \cdot)||_1 \leq C\delta$. By choosing δ small enough, the claim follows. If ω is bad on level 4 and $\mathcal{B}_L \subset V_{L/2}$, we do not gain a factor δ from $||1_{\mathcal{B}_L}\Delta(y, \cdot)||_1$. However, thanks to our modified transition kernels, using (25), $||1_{\mathcal{B}_L}\Delta\breve{G}^g 1_{\mathcal{B}_L}(y, \cdot)||_1 \leq 1/5$ (recall that $\breve{G}^g \leq \hat{G}^g$ pointwise), so that (ii) follows in this case, too. □

Remark 4.2. All $\delta_0 > 0$ and L_0 appearing in the next sections are understood to be chosen in such a way that if we take $\delta \in (0, \delta_0]$ and $L \geq L_0$, then the conclusions of Lemmata 4.1, 4.4, 4.5 and Corollary 4.1 are valid.

5 Globally smoothed exits

The aim here is to establish the estimates for the smoothed difference $D^*_{L,\psi}$ which are required to propagate condition **C1** (δ, L). For the entire section, we choose $r = r_L$. We start with an auxiliary statement which will be of constant use.

Lemma 5.1. *Let $\psi \in \mathcal{M}_L$ and set $\Delta = 1_{V_L}(\hat{\Pi}^g_{L,r_L} - \hat{\pi}_{L,r_L})$. Then, for some $C > 0$,*

$$\sup_{x \in V_L} \sup_{z \in \mathbb{Z}^d} |\Delta\phi_{L,\psi}(x, z)| \leq C(\log L)^{-12}L^{-d}.$$

Proof: Using $\Delta\phi = \Delta\hat{\pi}\phi$ and the fact that $\Delta\hat{\pi}(x,\cdot)$ sums up to zero,

$$\begin{aligned}|\Delta\phi(x,z)| &= \left|\sum_{y\in V_L\cup\partial V_L}\Delta\hat{\pi}(x,y)\left(\phi(y,z)-\phi(x,z)\right)\right|\\ &\leq ||\Delta\hat{\pi}(x,\cdot)||_1\sup_{y:|\Delta\hat{\pi}(x,y)|>0}|\phi(y,z)-\phi(x,z)|.\end{aligned}$$

For $x\in V_L\setminus \mathrm{Sh}_L(2r_L)$, we have by definition $||\Delta\hat{\pi}(x,\cdot)||_1 \leq C(\log L)^{-9}$. Further, notice that $|\Delta\hat{\pi}(x,y)|>0$ implies $|y-x|\leq s_L$. Bounding $|\phi(y,z)-\phi(x,z)|$ by Lemma 3.5 (iii), the statement follows for those x. If $x\in \mathrm{Sh}_L(2r_L)$, we simply bound $||\Delta\hat{\pi}(x,\cdot)||_1$ by 2. Now we can restrict the supremum to those $y\in V_L$ with $|x-y|\leq 3r_L$, so the claim follows again from Lemma 3.5 (iii). □

5.1 Estimates on "goodified" environments

The following Lemma 5.2 compares the "goodified" smoothed exit distribution with that of simple random walk. In particular, it provides an estimate for $D^*_{L,\psi}$ on Good_L. Here we will work with the transition kernels $\hat{\Pi}_{L,r_L}$, $\hat{\Pi}^g_{L,r_L}$ and $\hat{\pi}_{L,r_L}$. For the goodified exit measure from V_L we write

$$\Pi^g_L = \mathrm{ex}_{V_L}\left(x,\cdot;\hat{\Pi}^g_{L,r}\right).$$

Lemma 5.2. *Assume* **A1**. *There exist $\delta_0>0$ and $L_0>0$ such that if $\delta\in(0,\delta_0]$ and $L\geq L_0$, then for $\psi\in\mathcal{M}_L$,*

$$\mathbb{P}\left(\sup_{x\in V_L}||(\Pi^g_L-\pi_L)\hat{\pi}_\psi(x,\cdot)||_1 \geq (\log L)^{-(9+1/6)}\right)\leq \exp\left(-(\log L)^{7/3}\right).$$

Proof: Clearly, the claim follows if we show

$$\sup_{x\in V_L}\sup_{z\in\mathbb{Z}^d}\mathbb{P}\left(|(\Pi^g-\pi)\hat{\pi}_\psi(x,z)|\geq(\log L)^{-(9+1/5)}L^{-d}\right)\leq\exp\left(-(\log L)^{5/2}\right). \tag{27}$$

Using the abbreviations $\phi=\pi\hat{\pi}_\psi$, $\Delta = 1_{V_L}(\hat{\Pi}^g-\hat{\pi})$, we start with the perturbation expansion

$$(\Pi^g-\pi)\hat{\pi}_\psi = \hat{G}^g\Delta\phi.$$

Set $S=\mathrm{Sh}_L(2L/(\log L)^2)$ and write

$$\hat{G}^g\Delta\phi = \hat{G}^g1_S\Delta\phi + \hat{G}^g1_{S^c}\Delta\phi. \tag{28}$$

Using $\hat{G}^g\preceq C\Gamma$, Lemma 4.3 (iv) (with $r=r_L$) and Lemma 5.1 yield the estimate

$$|\hat{G}^g1_S\Delta\phi(x,z)|\leq\sup_{x\in V_L}\hat{G}^g(x,S)\sup_{y\in V_L}|\Delta\phi(y,z)|\leq(\log L)^{-19/2}L^{-d}$$

for L large. It remains to bound $|\hat{G}^g1_{S^c}\Delta\phi(x,z)|$. With $B=V_L\setminus\mathrm{Sh}_L(2r_L)$,

$$\hat{G}^g = \hat{g}1_B\Delta\hat{G}^g + \hat{g}1_{B^c}\Delta\hat{G}^g + \hat{g}.$$

By replacing successively $\hat{G}^g$ in the first summand on the right-hand side,

$$\begin{aligned}\hat{G}^g 1_{S^c}\Delta\phi &= \left(\sum_{k=0}^{\infty}(\hat{g}1_B\Delta)^k\hat{g} + \sum_{k=0}^{\infty}(\hat{g}1_B\Delta)^k\hat{g}1_{B^c}\Delta\hat{G}^g\right)1_{S^c}\Delta\phi \\ &= F1_{S^c}\Delta\phi + F1_{B^c}\Delta\hat{G}^g 1_{S^c}\Delta\phi, \qquad (29)\end{aligned}$$

where $F = \sum_{k=0}^{\infty}(\hat{g}1_B\Delta)^k\hat{g}$. With $R = \sum_{k=1}^{\infty}(1_B\Delta)^k\hat{\pi}$, expansion (5) shows

$$F = \hat{g}\sum_{m=0}^{\infty}(R\hat{g})^m\sum_{k=0}^{\infty}(1_B\Delta)^k.$$

From the proof of Lemma 4.1 (ii) we learn that $|F| \preceq C\Gamma$. By Lemma 4.3 (iv), (v) and again Lemma 5.1, we see that for large L, uniformly in $x \in V_L$ and $z \in \mathbb{Z}^d$,

$$\begin{aligned}|F1_{B^c}\Delta\hat{G}^g 1_{S^c}\Delta\phi(x,z)| &\leq C\Gamma(x, \mathrm{Sh}_L(2r_L))\sup_{v\in\mathrm{Sh}_L(3r_L)}\Gamma(v, S^c\cap V_L)\sup_{w\in V_L}|\Delta\phi(w,z)| \\ &\leq (\log L)^{-11}L^{-d}.\end{aligned}$$

Thus, the second summand of (29) is harmless. However, with the first summand one has to be more careful. With $\xi = \hat{g}\sum_{k=0}^{\infty}(1_B\Delta)^k 1_{S^c}\Delta\phi$, we have

$$F1_{S^c}\Delta\phi = \xi + \hat{g}\sum_{m=0}^{\infty}(R\hat{g})^m R\xi = \xi + F1_B\Delta\hat{\pi}\xi.$$

Clearly, $|F1_B\Delta\hat{\pi}(x,y)| \leq C(\log L)^{-3}$, so it remains to estimate $\xi(y,z)$, uniformly in y and z. Set $N = N(L) = \lceil \log\log L\rceil$. For small δ, the summands of ξ with $k \geq N$ are readily bounded by

$$\begin{aligned}\sup_{y\in V_L}\sup_{z\in\mathbb{Z}^d}\sum_{k=N}^{\infty}|\hat{g}(1_B\Delta)^k 1_{S^c}\Delta\phi(y,z)| &\leq C(\log L)^6\sum_{k=N}^{\infty}\delta^k(\log L)^{-12}L^{-d} \\ &\leq (\log L)^{-10}L^{-d}.\end{aligned}$$

Now we look at the summands with $k < N$. Since the coarse grained walk cannot bridge a gap of length $L/(\log L)^2$ in less than N steps, we can drop the kernel 1_B. Defining $S' = \mathrm{Sh}_L\left(3L/(\log L)^2\right)$, we thus have

$$\hat{g}(1_B\Delta)^k 1_{S^c}\Delta\phi = \hat{g}1_{S'}\Delta^k 1_{S^c}\Delta\phi + \hat{g}1_{S'^c}\Delta^k 1_{S^c}\Delta\phi.$$

The first summand is bounded in the same way as $\hat{G}^g 1_S\Delta\phi$ from (28). Further, we can drop the kernel 1_{S^c} in the second summand. Therefore, (27) follows if we show

$$\sup_{x\in V_L}\sup_{z\in\mathbb{Z}^d}\mathbb{P}\left(\left|\sum_{k=1}^{N}\hat{g}1_{S'^c}\Delta^k\phi(x,z)\right| \geq \frac{1}{2}(\log L)^{-(9+1/5)}L^{-d}\right) \leq \exp\left(-(\log L)^{5/2}\right).$$

For $j \in \mathbb{Z}$, consider the interval $I_j = (jNs_L, (j+1)Ns_L] \subset \mathbb{Z}$. We divide $S'^c \cap V_L$ into subsets $W_{\mathbf{j}} = (S'^c \cap V_L) \cap (I_{j_1} \times \ldots \times I_{j_d})$, where $\mathbf{j} = (j_1, \ldots, j_d) \in \mathbb{Z}^d$. Let J be the set of those $\mathbf{j}$ for which $W_{\mathbf{j}} \neq \emptyset$. Then we can find a constant K depending only on the dimension and a disjoint partition of J into sets $J_1, \ldots, J_K$, such that for any $1 \leq l \leq K$,

$$\mathbf{j}, \mathbf{j}' \in J_l,\ \mathbf{j} \neq \mathbf{j}' \Longrightarrow \mathrm{d}(W_{\mathbf{j}}, W_{\mathbf{j}'}) > Ns_L. \tag{30}$$

For $x \in V_L$, $z \in \mathbb{Z}^d$, we set

$$\xi_{\mathbf{j}} = \xi_{\mathbf{j}}(x,z) = \sum_{y \in W_{\mathbf{j}}} \sum_{k=1}^{N} \hat{g}(x,y) \Delta^k \phi(y,z),$$

and further $t = t(d,L) = (1/2)(\log L)^{-(9+1/5)} L^{-d}$. Assume that we can prove

$$\left| \sum_{\mathbf{j} \in J} \mathbb{E}\left[\xi_{\mathbf{j}}\right] \right| \leq \frac{t}{2}. \tag{31}$$

Then

$$\mathbb{P}\left(\left| \sum_{\mathbf{j} \in J} \xi_{\mathbf{j}} \right| \geq t \right) \leq \mathbb{P}\left(\left| \sum_{\mathbf{j} \in J} \xi_{\mathbf{j}} - \mathbb{E}\left[\xi_{\mathbf{j}}\right] \right| \geq \frac{t}{2} \right) \leq K \max_{1 \leq l \leq K} \mathbb{P}\left(\left| \sum_{\mathbf{j} \in J_l} \xi_{\mathbf{j}} - \mathbb{E}\left[\xi_{\mathbf{j}}\right] \right| \geq \frac{t}{2K} \right).$$

Due to (30), the random variables $\xi_{\mathbf{j}} - \mathbb{E}\left[\xi_{\mathbf{j}}\right]$, $\mathbf{j} \in J_l$, are independent and centered. Hoeffding's inequality yields, with $||\xi_{\mathbf{j}}||_\infty = \sup_{\omega \in \Omega} |\xi_{\mathbf{j}}(\omega)|$,

$$\mathbb{P}\left(\left| \sum_{\mathbf{j} \in J_l} \xi_{\mathbf{j}} - \mathbb{E}\left[\xi_{\mathbf{j}}\right] \right| \geq \frac{t}{2K} \right) \leq 2 \exp\left(-c \frac{L^{-2d} (\log L)^{-(18+2/5)}}{\sum_{\mathbf{j} \in J_l} ||\xi_{\mathbf{j}}||_\infty^2} \right) \tag{32}$$

for some constant $c > 0$. In order to control the sup-norm of the $\xi_{\mathbf{j}}$, we use the estimates

$$\hat{g}(x, W_{\mathbf{j}}) \leq C\Gamma^{(2)}(x, W_{\mathbf{j}}) \leq \frac{CN^d s_L^d}{s_L^2 (s_L + \mathrm{d}(x, W_{\mathbf{j}}))^{d-2}} = CN^d \left(1 + \frac{\mathrm{d}(x, W_{\mathbf{j}})}{s_L} \right)^{2-d},$$

and, by Lemma 5.1 for $y \in W_{\mathbf{j}}$, $\left|\Delta^k \phi(y,z)\right| \leq C\delta^{k-1} k (\log L)^{-12} L^{-d}$. Altogether we arrive at

$$||\xi_{\mathbf{j}}||_\infty \leq C \left(1 + \frac{\mathrm{d}(x, W_{\mathbf{j}})}{s_L} \right)^{2-d} N^d (\log L)^{-12} L^{-d},$$

uniformly in z. If we put the last display into (32), we get, using $d \geq 3$ in the last line,

$$\begin{aligned} \mathbb{P}\left(\left| \sum_{\mathbf{j} \in J_l} \xi_{\mathbf{j}} - \mathbb{E}\left[\xi_{\mathbf{j}}\right] \right| \geq \frac{t}{2K} \right) &\leq 2 \exp\left(-c \frac{(\log L)^{6-2/5}}{N^4 \sum_{r=1}^{C(\log L)^3/N} r^{-d+3}} \right) \\ &\leq 2 \exp\left(-c \frac{(\log L)^{3-2/5}}{N^3} \right). \end{aligned}$$

It follows that for L large enough, uniformly in x and z,

$$\mathbb{P}\left(\left|\sum_{\mathbf{j}\in J}\xi_{\mathbf{j}}\right| \geq \frac{1}{2}(\log L)^{-(9+1/5)}L^{-d}\right) \leq \exp\left(-(\log L)^{5/2}\right).$$

It remains to prove (31). We have

$$\left|\sum_{\mathbf{j}\in J}\mathbb{E}\left[\xi_{\mathbf{j}}\right]\right| \leq \sum_{y\in S'^c}\hat{g}(x,y)\left|\sum_{y'\in V_L}\mathbb{E}\left[\sum_{k=1}^{N}\Delta^k\hat{\pi}(y,y')\right]\phi(y',z)\right|.$$

Now (31) follows from the estimates $\hat{g}(x,S'^c) \leq C(\log L)^6$ and

$$\sup_{y\in S'^c}\left|\sum_{y'\in V_L}\mathbb{E}\left[\sum_{k=1}^{N}\Delta^k\hat{\pi}(y,y')\right]\phi(y',z)\right| \leq C(\log L)^{-18}L^{-d},$$

which in turn follows from Proposition 3.1 applied to $\nu(\cdot) = \mathbb{E}\left[\sum_{k=1}^{N}\Delta^k\hat{\pi}(y,y+\cdot)\right]$. □

Remark 5.1. The reader should notice that for $y \in S'^c$, the signed measure ν fulfills the requirements (i) and (ii) of Proposition 3.1. Indeed, after $N = \lceil\log\log L\rceil$ steps away from y, the coarse grained walks are still in the interior part of V_L, where the coarse graining radius did not start to shrink. Due to **A1**, we thus deduce that (i) and (ii) hold true for the signed measure $\mathbb{E}[\sum_{k=1}^{N}(1_{V_L}(\hat{\Pi}-\hat{\pi}))^k\hat{\pi}(y,y+\cdot)]$. Replacing $\hat{\Pi}$ by $\hat{\Pi}^g$ does not destroy the symmetries of this measure, so that Proposition 3.1 can be applied to ν.

5.2 Estimates in the presence of bad points

In the following lemma, we estimate $D^*_{L,\psi}$ on environments with bad points. We work with the modified kernels $\breve{\Pi}$, $\breve{\Pi}^g$, $\breve{\pi}$ from Section 4.4. Recall that the exit measures under these kernels do not change, e.g. $\Pi^g_L = \mathrm{ex}_{V_L}(x,\cdot;\breve{\Pi}^g_{L,r})$. Again, we make the choice $r = r_L$ for the coarse graining scheme.

Lemma 5.3. *In the setting of Lemma 5.2, for $i = 1,2,3,4$,*

$$\mathbb{P}\left(\sup_{x\in V_{L/5}}||(\Pi_L-\pi_L)\hat{\pi}_\psi(x,\cdot)||_1 > (\log L)^{-9+9(i-1)/4};\ \mathrm{OneBad}_L^{(i)}\right) \leq \exp\left(-(\log L)^{7/3}\right).$$

Proof: By the triangle inequality,

$$||(\Pi-\pi)\hat{\pi}_\psi(x,\cdot)||_1 \leq ||(\Pi-\Pi^g)\hat{\pi}_\psi(x,\cdot)||_1 + ||(\Pi^g-\pi)\hat{\pi}_\psi(x,\cdot)||_1. \tag{33}$$

The second summand on the right is estimated by Lemma 5.2. For the first term we have, with $\Delta = 1_{V_L}(\breve{\Pi}-\breve{\Pi}^g)$,

$$(\Pi-\Pi^g)\hat{\pi}_\psi = \breve{G}^g 1_{\mathcal{B}_L}\Delta\Pi\hat{\pi}_\psi.$$

Note that since we are on OneBad_L, the set $\mathcal{B}_L$ is contained in a small region. First assume that $\mathcal{B}_L \subset \mathrm{Sh}_L(L/(\log L)^{10})$. Then $\sup_{x\in V_{L/5}} \breve{G}^g(x,\mathcal{B}_L) \leq C(\log L)^{-10}$ by Corollary 4.1, which bounds the first summand of (33). Next assume ω bad on level 4 and $\mathcal{B}_L \not\subset V_{L/2}$. Then $\sup_{x\in V_{L/5}} \breve{G}^g(x,\mathcal{B}_L) \leq C(\log L)^{-3}$ by the same corollary, which is good enough for this case.

It remains to consider the cases ω bad on level at most 3 with $\mathcal{B}_L \not\subset \mathrm{Sh}_L(L/(\log L)^{10})$, or ω bad on level 4 with $\mathcal{B}_L \subset V_{L/2}$. We put $\phi = \pi\hat{\pi}_\psi$ and expand

$$\begin{aligned}(\Pi - \Pi^g)\hat{\pi}_\psi = \left(\breve{G}^g 1_{\mathcal{B}_L}\Delta\Pi\right)\hat{\pi}_\psi &= \sum_{k=1}^{\infty}\left(\breve{G}^g 1_{\mathcal{B}_L}\Delta\right)^k \Pi^g\hat{\pi}_\psi \\ &= \sum_{k=1}^{\infty}\left(\breve{G}^g 1_{\mathcal{B}_L}\Delta\right)^k \phi + \sum_{k=1}^{\infty}\left(\breve{G}^g 1_{\mathcal{B}_L}\Delta\right)^k (\Pi^g - \pi)\hat{\pi}_\psi \\ &= F_1 + F_2.\end{aligned}$$

By Corollary 4.1,

$$\begin{aligned}||F_1(x,\cdot)||_1 &\leq \sum_{k=0}^{\infty}||(\breve{G}^g 1_{\mathcal{B}_L}\Delta)^k(x,\cdot)||_1 \sup_{v\in V_L}\breve{G}^g(v,\mathcal{B}_L)\sup_{w\in\mathcal{B}_L}||\Delta\phi(w,\cdot)||_1 \\ &\leq C\sup_{w\in\mathcal{B}_L}||\Delta\phi(w,\cdot)||_1.\end{aligned}$$

Proceeding as in Lemma 5.1,

$$||\Delta\phi(w,\cdot)||_1 \leq ||\Delta\hat{\pi}(w,\cdot)||_1 \sup_{w'\colon|\Delta\hat{\pi}(w,w')|>0}||\phi(w',\cdot)-\phi(w,\cdot)||_1.$$

If ω is not bad on level 4, we have on $\mathcal{B}_L$ the equality $\Delta = \hat{\Pi} - \hat{\pi}$. Since $\mathcal{B}_L \cap \mathrm{Sh}_L(2r_L) = \emptyset$, this gives $\sup_{w\in\mathcal{B}_L}||\Delta\hat{\pi}(w,\cdot)||_1 \leq C(\log L)^{-9+9i/4}$ for every $i = 1,2,3,4$. For the second factor in the last display, Lemma 3.5 (iii) yields the bound $C(\log L)^{-3}$. We arrive at $||F_1(x,\cdot)||_1 \leq C(\log L)^{-12+9i/4}$. For F_2, we obtain once more with Corollary 4.1,

$$||F_2(x,\cdot)||_1 \leq C\sup_{y\in V_L}||(\Pi^g - \pi)\hat{\pi}_\psi(y,\cdot)||_1.$$

This term is again estimated by Lemma 5.2, and the lemma is proved. □

6 Non-smoothed and locally smoothed exits

Here, we aim at bounding the total variation distance of the exit measures without additional smoothing (Lemma 6.1), as well as in the case where a kernel of constant smoothing radius s is added (Lemma 6.2). We use the transition kernels $\breve{\Pi}, \breve{\Pi}^g$ and $\breve{\pi}$.

Throughout this section, we work with constant parameter r. We always assume L large enough such that $r < r_L$. The right choice of r depends on the deviations δ and η we are shooting for and will become clear from the proofs. In either case, we choose $r \geq r_0$, where r_0 is the constant from Section 2.4. The value of r will then also influence the choice of the perturbation ε_0 in Lemma 6.1 and the smoothing radius l in Lemma 6.2, respectively.

We recall the partition of bad points into the sets $\mathcal{B}_L$, $\mathcal{B}_{L,r}$, $\mathcal{B}^{\partial}_{L,r}$, $\mathcal{B}^{\star}_{L,r}$ and the classification of environments into Good_L, OneBad_L and $\mathrm{BdBad}_{L,r}$ from Section 2.

The bounds for $\mathrm{ManyBad}_L$ (Lemma 2.1) and for $\mathrm{BdBad}_{L,r}$ (Lemma 2.2) ensure that we may restrict ourselves to environments $\omega \in \mathrm{OneBad}_L \cap (\mathrm{BdBad}_{L,r})^c$. For such environments, we introduce two disjoint random sets $Q^1_{L,r}(\omega)$, $Q^2_{L,r}(\omega) \subset V_L$ as follows:

- If $\mathcal{B}_L(\omega) \subset V_{L/2}$, set $Q^1_{L,r}(\omega) = \mathcal{B}_L(\omega)$ and $Q^2_{L,r}(\omega) = \mathcal{B}^{\partial}_{L,r}(\omega)$.
- If $\mathcal{B}_L(\omega) \not\subset V_{L/2}$, set $Q^1_{L,r}(\omega) = \emptyset$ and $Q^2_{L,r}(\omega) = \mathcal{B}^{\star}_{L,r}(\omega)$.

Of course, on Good_L, we have $Q^1_{L,r}(\omega) = \emptyset$ and $Q^2_{L,r}(\omega) = \mathcal{B}^{\partial}_{L,r}(\omega)$.

Lemma 6.1. *There exists $\delta_0 > 0$ such that if $\delta \in (0, \delta_0]$, there exist $\varepsilon_0 = \varepsilon_0(\delta) > 0$ and $L_0 = L_0(\delta) > 0$ with the following property: If $\varepsilon \leq \varepsilon_0$ and $L_1 \geq L_0$, then* $\mathbf{A0}(\varepsilon)$, $\mathbf{C1}(\delta, L_1)$ *imply that for $L_1 \leq L \leq L_1(\log L_1)^2$,*

$$\mathbb{P}\left(\sup_{x \in V_{L/5}} ||(\Pi_L - \pi_L)(x, \cdot)||_1 > \delta\right) \leq \exp\left(-\frac{9}{5}(\log L)^2\right).$$

Proof: We choose $\delta_0 > 0$ according to Remark 4.2 and take $\delta \in (0, \delta_0]$. The right choice of ε_0 and L_0 will be clear from the course of the proof. From Lemmata 2.1 and 2.2 we learn that if we take L_1 large enough and L with $L_1 \leq L \leq L_1(\log L_1)^2$, then under $\mathbf{C1}(\delta, L_1)$

$$\mathbb{P}\left(\mathrm{ManyBad}_L \cup \mathrm{BdBad}_{L,r}\right) \leq \exp\left(-\frac{9}{5}(\log L)^2\right).$$

Therefore, the claim follows if we show that on $\mathrm{OneBad}_L \cap (\mathrm{BdBad}_{L,r})^c$, we have for all sufficiently small ε and all large L, $x \in V_{L/5}$,

$$||(\Pi - \pi)(x, \cdot)||_1 \leq \delta.$$

Let $\omega \in \mathrm{OneBad}_L \cap (\mathrm{BdBad}_{L,r})^c$. We use the partition of $\mathcal{B}^{\star}_{L,r}$ into the sets Q^1, Q^2 described above. With $\Delta = 1_{V_L}(\breve{\Pi} - \breve{\Pi}^g)$, we have inside V_L

$$\Pi = \breve{G}^g 1_{Q^1} \Delta \Pi + \breve{G}^g 1_{Q^2} \Delta \Pi + \Pi^g.$$

By replacing successively Π in the first summand on the right-hand side, we arrive at

$$\Pi = \sum_{k=0}^{\infty} \left(\breve{G}^g 1_{Q^1} \Delta\right)^k \Pi^g + \sum_{k=0}^{\infty} \left(\breve{G}^g 1_{Q^1} \Delta\right)^k \breve{G}^g 1_{Q^2} \Delta \Pi.$$

Since with $\Delta' = 1_{V_L}(\breve{\Pi}^g - \breve{\pi})$, $\Pi^g = \pi + \breve{G}^g \Delta' \pi$, we obtain

$$\begin{aligned} &\Pi - \pi \\ &= \sum_{k=1}^{\infty} \left(\breve{G}^g 1_{Q^1} \Delta\right)^k \pi + \sum_{k=0}^{\infty} \left(\breve{G}^g 1_{Q^1} \Delta\right)^k \breve{G}^g 1_{Q^2} \Delta \Pi + \sum_{k=0}^{\infty} \left(\breve{G}^g 1_{Q^1} \Delta\right)^k \breve{G}^g \Delta' \pi \\ &= F_1 + F_2 + F_3. \end{aligned} \tag{34}$$

We will now prove that each of the three parts F_1, F_2, F_3 is bounded by $\delta/3$. If $Q^1 \neq \emptyset$, then $Q^1 = \mathcal{B}_L \subset V_{L/2}$ and $Q^2 = \mathcal{B}^{\partial}_{L,r}$. Using Corollary 4.1 in the second and Lemma 3.1 (ii) in the third inequality,

$$\begin{aligned} ||F_1(x,\cdot)||_1 &\leq \sum_{k=0}^{\infty} ||(\breve{G}^g 1_{\mathcal{B}_L}\Delta)^k(x,\cdot)||_1 \sup_{y\in V_L} \breve{G}^g(y,\mathcal{B}_L) \sup_{z\in\mathcal{B}_L} ||\Delta\pi(z,\cdot)||_1 \\ &\leq C \sup_{z\in V_{L/2}} ||\Delta\pi(z,\cdot)||_1 \leq C(\log L)^{-3} \leq C(\log L_0)^{-3} \leq \delta/3 \end{aligned} \tag{35}$$

for $L_0 = L_0(\delta)$ large enough, $L \geq L_0$. Regarding F_2, we have in the case $Q^1 \neq \emptyset$ by Corollary 4.1 (ii)

$$||F_2(x,\cdot)||_1 \leq C \sup_{y\in V_{2L/3}} \breve{G}^g(y,\mathcal{B}^{\partial}_{L,r}) \leq C(\log r)^{-1/2}.$$

On the other hand, if $Q^1 = \emptyset$, then $\mathcal{B}_L$ is outside $V_{L/3}$, so that by Corollary 4.1 (i), (ii)

$$||F_2(x,\cdot)||_1 \leq 2\breve{G}^g(x,\mathcal{B}^{\partial}_{L,r}\cup\mathcal{B}_L) \leq C\left((\log L)^{-3} + (\log r)^{-1/2}\right).$$

Altogether, for all $L \geq L_0$, by choosing $r = r(\delta)$ and $L_0 = L_0(\delta, r)$ large enough,

$$||F_2(x,\cdot)||_1 \leq C\left((\log L_0)^{-3} + (\log r)^{-1/2}\right) \leq \delta/3. \tag{36}$$

It remains to handle F_3. Once again with Corollary 4.1 (iii) for some $C_3 > 0$,

$$||F_3(x,\cdot)||_1 \leq C_3 \sup_{y\in V_{2L/3}} \left|\left|\breve{G}^g\Delta'\pi(y,\cdot)\right|\right|_1.$$

We have by definition of Δ',

$$\breve{G}^g\Delta'\pi = \breve{G}^g 1_{V_L\setminus\mathcal{B}^{\star}_{L,r}}\Delta'\pi + \breve{G}^g 1_{\mathcal{B}_L}\Delta'\pi, \tag{37}$$

and Δ' vanishes on $\mathcal{B}_L$ except for the case ω bad on level 4 with $\mathcal{B}_L \subset V_{L/2}$. In this case, we use Corollary 4.1 (i) and Lemma 3.1 (ii) to obtain

$$||\breve{G}^g 1_{\mathcal{B}_L}\Delta'\pi(y,\cdot)||_1 \leq C(\log L)^{-3} \leq C_3^{-1}\delta/12 \tag{38}$$

for L_0 large enough, $L \geq L_0$. Concerning the first term of (37), we note that on $V_L\setminus\mathcal{B}^{\star}_{L,r}$, $\Delta'\pi = (\hat{\Pi} - \hat{\pi})\hat{\pi}\pi$. Therefore, if $z \in V_L\setminus\left(\mathcal{B}^{\star}_{L,r}\cup\mathrm{Sh}_L(2r_L)\right)$, we obtain $||\Delta'\pi(z,\cdot)||_1 \leq C(\log L)^{-9}$. Since $\breve{G}^g(y,V_L) \leq C(\log L)^6$, it follows that

$$\sup_{y\in V_{2L/3}} \left|\left|\breve{G}^g 1_{V_L\setminus\left(\mathcal{B}^{\star}_{L,r}\cup\mathrm{Sh}_L(2r_L)\right)}\Delta'\pi(y,\cdot)\right|\right|_1 \leq C(\log L)^{-3} \leq C_3^{-1}\delta/12 \tag{39}$$

for L large. Recall the definition of the layers Λ_j from Section 2.4. For $z \in \Lambda_j\setminus\mathcal{B}^{\star}_{L,r}$, $1 \leq j \leq J_1$, we have $||\Delta'\pi(z,\cdot)||_1 \leq C(\log r + j)^{-9}$. By Lemma 4.3 (iii), $\breve{G}^g(y,\Lambda_j) \leq C$ for some constant C, independent of r and j. Therefore,

$$\sup_{y\in V_{2L/3}} \left|\left|\breve{G}^g 1_{\bigcup_{j=1}^{J_1}\Lambda_j\setminus\mathcal{B}^{\star}_{L,r}}\Delta'\pi(y,\cdot)\right|\right|_1 \leq C(\log r)^{-8} \leq C_3^{-1}\delta/12, \tag{40}$$

if r is chosen large enough. Finally, for the first layer Λ_0, there is a constant C_0 satisfying

$$\sup_{y\in V_{2L/3}} \left|\left|\breve{G}^g 1_{\Lambda_0\setminus\mathcal{B}^\star_{L,r}}\Delta'\pi(y,\cdot)\right|\right|_1 \le C_0 \sup_{z\in\Lambda_0} ||\Delta'(z,\cdot)||_1.$$

Now we take $\varepsilon_0 = \varepsilon_0(\delta, r)$ small enough such that for $\varepsilon \le \varepsilon_0$, $\sup_{z\in\Lambda_0}||\Delta'(z,\cdot)||_1 \le C_0^{-1}C_3^{-1}\delta/12$. We have shown that $||F_3(x,\cdot)||_1 \le \delta/3$, and the lemma is proven. □

Remark 6.1. As the proof shows, we do not have to assume $\mathbf{C1}(\delta, L_1)$ for the desired deviation δ. We could instead assume $\mathbf{C1}(\delta', L_1)$ for some $0 < \delta' \le \delta_0$. However, L_1 has to be larger than L_0, which depends on δ. This observation will be useful in the next lemma.

Lemma 6.2. *There exists $\delta_0 > 0$ with the following property: For each $\eta > 0$, there exist a smoothing radius $l_0 = l_0(\eta)$ and $L_0 = L_0(\eta)$ such that if $L_1 \ge L_0$, $l \ge l_0$ and $\mathbf{C1}(\delta, L_1)$ holds for some $\delta \in (0, \delta_0]$, then for $L_1 \le L \le L_1(\log L_1)^2$ and $\psi \equiv l$,*

$$\mathbb{P}\left(\sup_{x\in V_{L/5}} ||(\Pi_L - \pi_L)\hat{\pi}_\psi(x,\cdot)||_1 > \eta\right) \le \exp\left(-\frac{9}{5}(\log L)^2\right).$$

Proof: The proof is based on a modification of the computations in the foregoing lemma. Let δ_0 be as in Lemma 6.1. We fix an arbitrary $0 < \delta \le \delta_0$ and assume $\mathbf{C1}(\delta, L_1)$ for some $L_1 \ge L_0$, where $L_0 = L_0(\eta)$ will be chosen later. In the following, "good" and "bad" is always to be understood with respect to δ. Again, for $L_1 \le L \le L_1(\log L_1)^2$,

$$\mathbb{P}\left(\mathrm{ManyBad}_L \cup \mathrm{BdBad}_{L,r}\right) \le \exp\left(-\frac{9}{5}(\log L)^2\right).$$

For $\omega \in \mathrm{OneBad}_L \cap (\mathrm{BdBad}_{L,r})^c$, we use the splitting (34) of $\Pi - \pi$ into the parts F_1, F_2, F_3. For the summands F_1 and F_2, we do not need the additional smoothing by $\hat{\pi}_\psi$, since by (35)

$$||F_1(x,\cdot)||_1 \le C(\log L)^{-3} \le \eta/3,$$

and by (36)

$$||F_2(x,\cdot)||_1 \le C\left((\log L)^{-3} + (\log r)^{-1/2}\right) \le \eta/3,$$

if $L \ge L_0$ and r, L_0 are chosen large enough, depending on d and η. We turn to F_3. With (38), (39) and (40) we have (recall that $\Delta' = 1_{V_L}(\breve{\Pi}^g - \breve{\pi})$)

$$\begin{aligned} ||F_3\hat{\pi}_s(x,\cdot)||_1 &\le C\left(\sup_{y\in V_{2L/3}} \left|\left|\breve{G}^g 1_{V_L\setminus\Lambda_0}\Delta'\pi(y,\cdot)\right|\right|_1 + \sup_{z\in\Lambda_0}||\Delta'\pi\hat{\pi}_\psi(z,\cdot)||_1\right) \\ &\le C\left((\log L)^{-3} + (\log r)^{-8} + \sup_{z\in\Lambda_0}||\Delta'\pi\hat{\pi}_\psi(z,\cdot)||_1\right) \\ &\le \eta/6 + C_1 \sup_{z\in\Lambda_0}||\Delta'\pi\hat{\pi}_\psi(z,\cdot)||_1, \end{aligned} \tag{41}$$

if $L \geq L_0$ and r, L_0 are sufficiently large. Regarding the second summand of (41), set $m = 3r$ and define for $K \in \mathbb{N}$

$$\vartheta_K(z) = \min\{n \in \mathbb{N} : |X_n^z - z| > Km\} \in [0, \infty],$$

where X_n^z denotes simple random walk with start in z. By the invariance principle for simple random walk, we can choose K so large such that

$$\max_{z \in V_L :\, \mathrm{d}_L(z) \leq m} \mathrm{P}_z\left(\vartheta_K(z) \leq \tau_L\right) \leq \frac{\eta}{24 C_1}$$

uniformly in $L \geq L_0$, where C_1 is the constant from (41). If $z \in \Lambda_0$, $z' \in V_L \cup \partial V_L$ with $\Delta'(z, z') \neq 0$, we have $\mathrm{d}_L(z') \leq m$ and $|z - z'| \leq m$. Thus, using Lemma 10.2 (iii) of the appendix with $\psi \equiv l$,

$$\begin{aligned}
&C_1 \sup_{z \in \Lambda_0} ||\Delta' \pi \hat{\pi}_\psi(z, \cdot)||_1 \\
&\leq \; C_1 \sup_{z \in \Lambda_0} \Bigg\| \sum_{\substack{z' \in V_L \cup \partial V_L : \\ \Delta'(z,z') \neq 0}} \Delta'(z, z') \\
&\qquad \left(\sum_{\substack{w \in \partial V_L : \\ |z' - w| > Km}} \pi(z', w) + \sum_{\substack{w \in \partial V_L : \\ |z' - w| \leq Km}} \pi(z', w) \right) \left(\hat{\pi}_\psi(w, \cdot) - \hat{\pi}_\psi(z, \cdot)\right) \Bigg\|_1 \\
&\leq \; \frac{\eta}{6} + C(K+1)m \frac{\log l}{l} \leq \eta/3,
\end{aligned}$$

if we choose $l = l(\eta, r)$ large enough. This proves the lemma. □

7 Proofs of the main results on exit laws

Proof of Proposition 1.1: We take δ_0 small enough and, for $\delta \leq \delta_0$, we choose $L_0 = L_0(\delta)$ large enough according to Remark 4.2 and the statements of Sections 5, 6. (ii) is a consequence of Lemma 6.2, so we have to prove (i). Let $L_1 \geq L_0$, and assume that $\mathbf{C1}(\delta, L_1)$ holds. Then, for $i = 1, 2, 3$ and $L_1 \leq L \leq L_1 (\log L_1)^2$, $\psi \in \mathcal{M}_L$, using Lemma 2.1,

$$\begin{aligned}
b_i(L, \psi, \delta) &\leq \mathbb{P}\left(D^*_{L,\psi} > (\log L)^{-9+9(i-1)/4}\right) \\
&\leq \mathbb{P}\left(\mathrm{ManyBad}_L\right) + \mathbb{P}\left(D^*_{L,\psi} > (\log L)^{-9+9(i-1)/4};\ \mathrm{OneBad}_L\right) \\
&\leq \exp\left(-\frac{19}{10}(\log L)^2\right) + \mathbb{P}\left(D^*_{L,\psi} > (\log L)^{-9+9(i-1)/4};\ \mathrm{OneBad}_L\right).
\end{aligned}$$

For the last summand, we have by Lemmata 5.2, 5.3, under $\mathbf{C1}(\delta, L_1)$,

$$\begin{aligned}
&\mathbb{P}\left(D^*_{L,\psi} > (\log L)^{-9+9(i-1)/4};\ \mathrm{OneBad}_L\right)\\
&\le\ \mathbb{P}\left(D^*_{L,\psi} > (\log L)^{-9};\ \mathrm{Good}_L\right) + \sum_{j=1}^{4} \mathbb{P}\left(D^*_{L,\psi} > (\log L)^{-9+9(i-1)/4};\ \mathrm{OneBad}_L^{(j)}\right)\\
&\le\ \exp\left(-(\log L)^{7/3}\right) + \sum_{j=1}^{i} \mathbb{P}\left(D^*_{L,\psi} > (\log L)^{-9+9(i-1)/4};\ \mathrm{OneBad}_L^{(j)}\right)\\
&\quad + \sum_{j=i+1}^{4} \mathbb{P}\left(\mathrm{OneBad}_L^{(j)}\right)\\
&\le\ 4\ \exp\left(-(\log L)^{7/3}\right) + CL^d s_L^d \exp\left(-\left((3+i+1)/4\right)\left(\log(r_L/20)\right)^2\right).
\end{aligned}$$

Therefore, by enlarging L if necessary,

$$\mathbb{P}\left(D^*_{L,\psi} > (\log L)^{-9+9(i-1)/4};\ \mathrm{OneBad}_L\right) \le \frac{1}{8}\exp\left(-\left((3+i)/4\right)(\log L)^2\right),$$

and

$$b_i(L,\psi,\delta) \le \frac{1}{4}\exp\left(-\left((3+i)/4\right)(\log L)^2\right).$$

For the case $i = 4$, notice that

$$b_4(L,\psi,\delta) \le \mathbb{P}\left(D^*_{L,\psi} > (\log L)^{-9/4}\right) + \mathbb{P}\left(D^*_L > \delta\right).$$

The first summand can be estimated as the corresponding terms in the case $i = 1, 2, 3$, while for the last term we use Lemma 6.1. $\square$

As Theorem 1.1 now follows immediately, we turn to the proof of the local estimates. Here, the results from Section 4 play again a key role.

Proof of Theorem 1.2: As usual, we mostly drop L as index, so always $\pi = \pi_L$, $\hat{\Pi} = \hat{\Pi}_L$ and so on. For the whole proof, we let $r = r_L$. Choose δ_0 and L_0 as in Proposition 1.1. Recall the definition of Good_L from Section 2. By Proposition 1.1, we find δ, $\varepsilon > 0$ and $L_0 > 0$ such that under $\mathbf{A0}(\varepsilon)$ and $\mathbf{A1}$, condition $\mathbf{C1}(\delta, L)$ holds true for all $L \ge L_0$. We put $A_L = \mathrm{Good}_L$ and note that similar to Lemma 2.1, if $L \ge L_0$,

$$\mathbb{P}(A_L^c) \le \exp\left(-(1/2)(\log L)^2\right).$$

For the rest of the proof, take $\omega \in A_L$. On such environments, $\hat{G}$ equals $\hat{G}^g$ by our choice $r = r_L$. Now let us prove part (i). Observe that W_t can be covered by $K|W_t|r^{-(d-1)}$ many neighborhoods $V_{3r}(y)$, $y \in \mathrm{Sh}_L(r)$, as defined in Section 4.2, where K depends on the dimension only. In particular, $\Gamma(x, W_t) \le C(t/L)^{d-1}$. Applying Lemma 4.1 (ii), we deduce that

$$\Pi_L(x, W_t) = \hat{G}^g(x, W_t) \le C(t/L)^{d-1}.$$

From Lemma 3.1 (i) we know that if $x \in V_{\eta L}$, then for some constant $c = c(d, \eta)$,

$$\pi(x, z) \ge cL^{-(d-1)}.$$

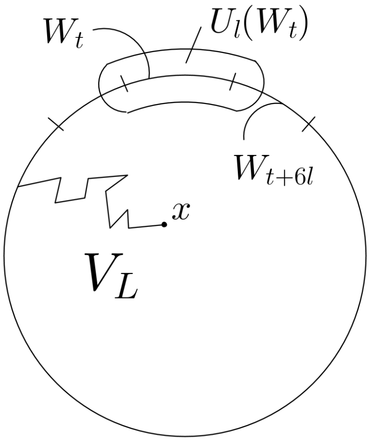

Figure 5: On the proof of Theorem 1.2 (ii). There, $t \geq L/(\log L)^6 > l = L/(\log L)^{17/2}$. If the walk exits V_L through $\partial V_L \backslash W_{t+6l}$, it cannot enter $U_l(W_t)$ in one step with $\hat{\pi}_l$.

Together with the preceding equation, this shows (i).
(ii) Set $l = (\log L)^{13/2} r$ and consider the smoothing kernel $\hat{\pi}_\psi$ with $\psi \equiv l \in \mathcal{M}_l$. Let

$$U_l(W_t) = \{y \in \mathbb{Z}^d : \mathrm{d}(y, W_t) \leq 2l\}.$$

We claim that

$$\begin{aligned} \Pi(x, W_t) - \pi(x, W_{t+6l}) &\leq (\Pi - \pi)\, \hat{\pi}_\psi(x, U_l(W_t)), &(42)\\ \pi(x, W_{t-6l}) - \Pi(x, W_t) &\leq (\pi - \Pi)\, \hat{\pi}_\psi(x, U_l(W_{t-6l})). &(43) \end{aligned}$$

Concerning the first inequality,

$$\Pi\hat{\pi}_\psi(x, U_l(W_t)) \geq \sum_{y \in W_t} \Pi(x, y)\hat{\pi}_\psi(y, U_l(W_t)) = \Pi(x, W_t),$$

since $\hat{\pi}_\psi(y, U_l(W_t)) = 1$ for $y \in W_t$. Also,

$$\pi\hat{\pi}_\psi(x, U_l(W_t)) = \sum_{y \in W_{t+6l}} \pi(x, y)\hat{\pi}_\psi(y, U_l(W_t)) \leq \pi(x, W_{t+6l}),$$

since $\hat{\pi}_\psi(y, U_l(W_t)) = 0$ for $y \in \partial V_L \backslash W_{t+6l}$. This proves (42), while (43) is entirely similar. In the remainder of this proof, we often write $|F|(x, y)$ for $|F(x, y)|$. If we show

$$|(\pi - \Pi)\, \hat{\pi}_\psi|\, (x, U_l(W_t)) \leq O\left((\log L)^{-5/2}\right) \pi(x, W_t), \tag{44}$$

then by (42),

$$\begin{aligned} \Pi(x, W_t) &\leq \pi(x, W_{t+6l}) + O((\log L)^{-5/2})\pi(x, W_t)\\ &= \pi(x, W_t) + \pi(x, W_{t+6l} \backslash W_t) + O((\log L)^{-5/2})\pi(x, W_t)\\ &= \pi(x, W_t)\left(1 + O\left(\max\{l/t, (\log L)^{-5/2}\}\right)\right)\\ &= \pi(x, W_t)\left(1 + O((\log L)^{-5/2})\right). \end{aligned}$$

On the other hand, by (43) and still assuming (44),

$$\Pi(x, W_t) \geq \pi(x, W_{t-6l}) - O((\log L)^{-5/2})\pi(x, W_t) = \pi(x, W_t)\left(1 - O((\log L)^{-5/2})\right),$$

so that indeed

$$\Pi(x, W_t) = \pi(x, W_t)\left(1 + O\left((\log L)^{-5/2}\right)\right),$$

provided we prove (44). In that direction, set $B = V_L \setminus \mathrm{Sh}_L(5r)$ and write, with $\Delta = 1_{V_L}(\hat{\Pi}^g - \hat{\pi})$,

$$(\pi - \Pi)\hat{\pi}_\psi = \hat{G}^g \Delta \pi \hat{\pi}_\psi = \hat{G}^g 1_B \Delta \pi \hat{\pi}_\psi + \hat{G}^g 1_{\mathrm{Sh}_L(5r)} \Delta \pi \hat{\pi}_\psi. \tag{45}$$

Looking at the first summand we have

$$|\hat{G}^g 1_B \Delta \pi \hat{\pi}_\psi|(x, U_l(W_t)) \leq (\hat{G}^g 1_B |\Delta \hat{\pi}| \pi)(x, W_{t+6l}).$$

Following the proof of Proposition 4.4 (ii), we deduce

$$\hat{G}^g 1_B |\Delta \hat{\pi}| \Gamma(x, z) \leq C(\log L)^{-5/2} \Gamma(x, z).$$

Together with $\pi \preceq C\Gamma$ and $\pi(x, z) \geq c(d, \eta) L^{-(d-1)}$ this yields the bound

$$\hat{G}^g 1_B |\Delta \hat{\pi}| \pi(x, W_{t+6l}) \leq C(\log L)^{-5/2} \Gamma(x, W_t) \leq C(\log L)^{-5/2} \pi(x, W_t).$$

To obtain (44), it remains to handle the second summand of (45), i.e. we have to bound

$$|\hat{G}^g 1_{\mathrm{Sh}_L(5r)} \Delta \pi \hat{\pi}_\psi|(x, U_l(W_t)).$$

We abbreviate $S = \mathrm{Sh}_L(5r)$ and split into

$$\begin{aligned}
&\hat{G}^g 1_S \Delta \pi \hat{\pi}_\psi(x, U_l(W_t)) \\
&\quad = \sum_{y \in S} \hat{G}^g(x, y) \sum_{z \in \partial V_L} \Delta \pi(y, z)(\hat{\pi}_\psi(z, U_l(W_t)) - \hat{\pi}_\psi(y, U_l(W_t))) \\
&\quad = \sum_{y \in S} \hat{G}^g(x, y) \sum_{z \in W_{t+6l}} \Delta \pi(y, z)(\hat{\pi}_\psi(z, U_l(W_t)) - \hat{\pi}_\psi(y, U_l(W_t))) \\
&\qquad - \sum_{y \in S} \hat{G}^g(x, y) \sum_{z \in \partial V_L \setminus W_{t+6l}} \Delta \pi(y, z) \hat{\pi}_\psi(y, U_l(W_t)).
\end{aligned}$$

First note that since $\hat{\pi}_\psi(y, z') = 0$ if $|y - z'| > 2l$,

$$\begin{aligned}
&\left| \sum_{y \in S} \hat{G}^g(x, y) \sum_{z \in \partial V_L \setminus W_{t+6l}} \Delta \pi(y, z) \hat{\pi}_\psi(y, U_l(W_t)) \right| \\
&\quad \leq \ (\hat{G}^g 1_{U_{2l}(W_t) \cap S} |\Delta \pi|)(x, \partial V_L \setminus W_{t+6l}).
\end{aligned}$$

For $y \in U_{2l}(W_t) \cap S$, we apply Lemma 3.2 (iii) together with Lemma 3.4 and obtain

$$|\Delta \pi|(y, \partial V_L \setminus W_{t+6l}) \leq \sup_{y' :\, \mathrm{d}(y', U_{2l}(W_t) \cap S) \leq r} \pi(y', \partial V_L \setminus W_{t+6l}) \leq C\frac{r}{l} \leq C(\log L)^{-13/2}.$$

Since $\hat{G}^g \preceq \Gamma$ and $\pi(x,z) \geq cL^{-d-1}$, $\hat{G}^g(x, U_{2l}(W_t) \cap S) \leq C\pi(x, W_t)$, and thus

$$(\hat{G}^g 1_{U_{2l}(W_t)\cap S}|\Delta\pi|)(x, \partial V_L \backslash W_{t+6l}) \leq C(\log L)^{-13/2}\pi(x, W_t).$$

It remains to bound

$$\left| \sum_{y\in S} \hat{G}^g(x,y) \sum_{z\in W_{t+6l}} \Delta\pi(y,z)(\hat{\pi}_\psi(z, U_l(W_t)) - \hat{\pi}_\psi(y, U_l(W_t))) \right|.$$

Set $D^1(y) = \{z \in W_{t+6l} : |z-y| \leq l(\log l)^{-4}\}$. If $D^1(y) \neq \emptyset$, then $\mathrm{d}(y, W_t) \leq 7l$. Using Lemma 10.2 (iii) for the difference of the smoothing steps and the usual estimate for $\hat{G}^g$,

$$\begin{aligned} &\sum_{y\in S} \hat{G}^g(x,y) \sum_{z\in D^1(y)} |\Delta\pi(y,z)||\hat{\pi}_\psi(z, U_l(W_t)) - \hat{\pi}_\psi(y, U_l(W_t))| \\ &\leq \; C\frac{t^{d-1}}{L^{d-1}}(\log l)^{-3} \leq C(\log L)^{-5/2}\pi(x, W_t). \end{aligned}$$

The region $W_{t+6l}\backslash D^1(y)$ we split into $B_0(y) = \{z \in W_{t+6l} : |z-y| \in (l(\log l)^{-4}, t]\}$, and

$$B_i(y) = \{z \in W_{t+6l} : |z-y| \in (it, (i+1)t]\}, \quad i = 1, 2, \ldots, \lfloor 2L/t \rfloor.$$

Furthermore, let

$$S_i = \{y \in S : B_i(y) \neq \emptyset\}, \quad i = 0, 1, \ldots, \lfloor 2L/t \rfloor.$$

Then

$$\sum_{y\in S} \hat{G}^g(x,y) \sum_{z\in W_{t+6l}\backslash D^1(y)} |\Delta\pi(y,z)| \leq C \sum_{i=0}^{\lfloor 2L/t \rfloor} \hat{G}^g(x, S_i) \sup_{y\in S_i} |\Delta\pi|(y, B_i(y)).$$

If $i \geq 1$ and $y \in S_i$, then by Lemma 3.2 (iii)

$$|\Delta\pi|(y, B_i(y)) \leq \sup_{y':|y'-y|\leq r} \pi(y', B_i(y)) \leq C\frac{rt^{d-1}}{(it)^d} \leq C\frac{r}{i^d t},$$

while in the case $i = 0$, using the same lemma and additionally Lemma 3.4,

$$\begin{aligned} \sup_{y':|y'-y|\leq r} \pi(y', B_i(y)) \; &\leq \; C\, r \sum_{z\in\partial V_L} \frac{1}{((1/2)l\,(\log l)^{-4} + |y-z|)^d} \\ &\leq \; C\frac{r(\log l)^4}{l} \leq C(\log L)^{-5/2}. \end{aligned}$$

For the Green's function, we use the estimates

$$\hat{G}^g(x, S_0) \leq C\frac{t^{d-1}}{L^{d-1}}, \quad \hat{G}^g(x, \cup_{i\geq(1/10)L/t} S_i) \leq C,$$

while for $i = 1, 2, \ldots, \lfloor (1/10)L/t \rfloor$, it holds that $|S_i| \le Cr(it)^{d-2}t$, whence

$$\hat{G}^g(x, S_i) \le C \frac{i^{d-2}t^{d-1}}{L^{d-1}}.$$

Altogether, we obtain

$$\begin{aligned} & \sum_{i=0}^{\lfloor 2L/t \rfloor} \hat{G}^g(x, S_i) \sup_{y \in S_i} |\Delta\pi|(y, B_i(y)) \\ \le \; & C \left((\log L)^{-5/2} \frac{t^{d-1}}{L^{d-1}} + \left(\frac{r}{t} \frac{t^{d-1}}{L^{d-1}} \sum_{i=1}^{\lfloor (1/10)L/t \rfloor} \frac{1}{i^2} \right) + \frac{t^{d-1}}{L^{d-1}} \frac{r}{L} \right) \\ \le \; & C (\log L)^{-5/2} \frac{t^{d-1}}{L^{d-1}}. \end{aligned}$$

This finishes the proof of part (ii). □

Let us finally show how to obtain transience of the RWRE.

Proof of Corollary 1.1: Fix numbers $\rho \ge 3$, $\alpha \in (0, (4\rho)^{-1})$ to be specified below. With these parameters and $n \ge 1$, we set

$$q_{n,\alpha,\rho} = \hat{\pi}_\psi,$$

where $\psi = (m_x)_{x \in \mathbb{Z}^d}$ is chosen constant in x, namely $m_x = \alpha\rho^n$. Define

$$A_n = \bigcap_{|x| \le \rho^{n^{3/2}}} \bigcap_{t \in [\alpha\rho^n, 2\alpha\rho^n]} \left\{ D_{t,\psi}(x) \le (\log t)^{-9} \right\}.$$

By Proposition 1.1 (i), there exists $\varepsilon_0 > 0$ such that given $\varepsilon \in (0, \varepsilon_0]$, $\mathbf{A0}(\varepsilon)$ implies that for n large enough, we have

$$\mathbb{P}\left(A_n^c\right) \le C\alpha^d \rho^{(d+1)n^{3/2}} \exp\left(- \left(\log\left(\alpha\rho^n\right)\right)^2 \right).$$

Therefore, for any choice of α, ρ it holds that

$$\sum_{n=1}^{\infty} \mathbb{P}\left(A_n^c\right) < \infty,$$

whence by Borel-Cantelli

$$\mathbb{P}\left(\liminf_{n \to \infty} A_n \right) = 1. \tag{46}$$

We denote the coarse grained RWRE transition kernel by

$$Q_{n,\alpha,\rho}(x, \cdot) = \frac{1}{\alpha\rho^n} \int_{\mathbb{R}_+} \varphi\left(\frac{t}{\alpha\rho^n} \right) \Pi_{V_t(x)}(x, \cdot) \mathrm{d}t.$$

If n is large enough and $|x| \leq \rho^{n^{3/2}}$, we have on A_n

$$||(Q_{n,\alpha,\rho} - q_{n,\alpha,\rho})\, q_{n,\alpha,\rho}(x,\cdot)||_1 \leq (\log(\alpha\rho^n))^{-9} \leq C(\alpha,\rho)n^{-9}.$$

Now assume $|x| \leq \rho^n + 1$. For N fixed, n large and $\omega \in A_n$, it follows that for $1 \leq M \leq N$

$$\left|\left|\left((Q_{n,\alpha,\rho})^M - (q_{n,\alpha,\rho})^M\right) q_{n,\alpha,\rho}(x,\cdot)\right|\right|_1 \leq C(\alpha,\rho)Mn^{-9}. \tag{47}$$

For fixed ω, let $(\xi_k)_{k\geq 0}$ be the Markov chain running with transition kernel $Q_{n,\alpha,\rho}$. Clearly, $(\xi_k)_{k\geq 0}$ can be obtained by observing the basic RWRE $(X_k)_{k\geq 0}$ at randomized stopping times. Then

$$\begin{aligned}
&\mathrm{P}_{x,\omega}\left(\xi_{N-1} \in V_{\rho^{n+1}+2\alpha\rho^n}\right)\\
&\leq \left(Q_{n,\alpha,\rho}\right)^{N-1} q_{n,\alpha,\rho}\left(x, V_{\rho^{n+1}+4\alpha\rho^n}\right)\\
&\leq \left|\left|\left((Q_{n,\alpha,\rho})^{N-1} - (q_{n,\alpha,\rho})^{N-1}\right) q_{n,\alpha,\rho}(x,\cdot)\right|\right|_1 + \left(q_{n,\alpha,\rho}\right)^N\left(x, V_{2\rho^{n+1}}\right).
\end{aligned}$$

Using Proposition 4.1, we can find $N = N(\alpha,\rho) \in \mathbb{N}$, depending not on n, such that for any x with $|x| \leq \rho^n + 1$, it holds that $(q_{n,\alpha,\rho})^N(x, V_{2\rho^{n+1}}) \leq 1/10$. With (47), we conclude that for such x, $n \geq n_0(\alpha,\rho,N)$ large enough and $\omega \in A_n$,

$$\mathrm{P}_{x,\omega}\left(\xi_{N-1} \in V_{\rho^{n+1}+2\alpha\rho^n}\right) \leq C(\alpha,\rho)Nn^{-9} + 1/10 \leq 1/5. \tag{48}$$

On the other hand, if x is outside $V_{\rho^{n-1}+2\alpha\rho^n}$,

$$\begin{aligned}
&\mathrm{P}_{x,\omega}\left(\xi_M \in V_{\rho^{n-1}+2\alpha\rho^n} \text{ for some } 0 \leq M \leq N-1\right)\\
&\leq \sum_{M=1}^{N-1} \left(Q_{n,\alpha,\rho}\right)^M q_{n,\alpha,\rho}\left(x, V_{\rho^{n-1}+4\alpha\rho^n}\right)\\
&\leq \sum_{M=1}^{N-1} \left|\left|\left((Q_{n,\alpha,\rho})^M - (q_{n,\alpha,\rho})^M\right) q_{n,\alpha,\rho}(x,\cdot)\right|\right|_1 + \sum_{k=2}^{N} \left(q_{n,\alpha,\rho}\right)^k\left(x, V_{2\rho^{n-1}}\right).
\end{aligned}$$

If $\rho^n - 1 \leq |x|$, then $(q_{n,\alpha,\rho})^k(x, V_{2\rho^{n-1}}) = 0$ as long as $k \leq (1-3/\rho)/(2\alpha)$. By first choosing ρ large enough, then α small enough and estimating the higher summands again with Proposition 4.1, we deduce that for such x and all large n,

$$\sum_{k=1}^{\infty} \left(q_{n,\alpha,\rho}\right)^k\left(x, V_{2\rho^{n-1}}\right) \leq 1/10.$$

Together with (47), we have for large n, $\omega \in A_n$ and $\rho^n - 1 \leq |x| \leq \rho^n + 1$,

$$\mathrm{P}_{x,\omega}\left(\xi_M \in V_{\rho^{n-1}+2\alpha\rho^n} \text{ for some } 0 \leq M \leq N-1\right) \leq C(\alpha,\rho)N^2n^{-9} + 1/10 \leq 1/5. \tag{49}$$

Let B be the event that the walk $(\xi_k)_{k\geq 0}$ leaves $V_{\rho^{n+1}+2\alpha\rho^n}$ before reaching $V_{\rho^{n-1}+2\alpha\rho^n}$. From (48) and (49) we deduce that $\mathrm{P}_{x,\omega}(B) \geq 3/5$, provided n is large enough, $\omega \in A_n$

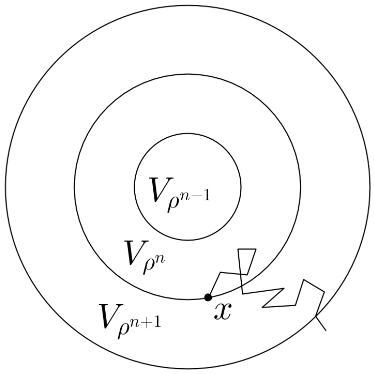

Figure 6: On a set of environments with mass 1, the RWRE started at any x with $|x| \geq \rho^n - 1$ leaves the ball $V_{\rho^{n+1}}$ before hitting $V_{\rho^{n-1}}$ with probability at least 3/5. This implies transience of the RWRE.

and $\rho^n - 1 \leq |x| \leq \rho^n + 1$. But on B, the underlying basic RWRE $(X_k)_{k \geq 0}$ clearly leaves $V_{\rho^{n+1}}$ before reaching $V_{\rho^{n-1}}$. Hence if $\omega \in \{\liminf A_n\}$, there exists $m_0 = m_0(\omega) \in \mathbb{N}$ such that

$$\mathrm{P}_{x,\omega}\left(\tau_{V_{\rho^{n+1}}} < T_{V_{\rho^{n-1}}}\right) \geq 3/5$$

for all $n \geq m_0$, x with $|x| \geq \rho^n - 1$ (of course, we may now drop the constraint $|x| \leq \rho^n + 1$). From this property, transience easily follows. Indeed, for $m, M, k \in \mathbb{N}$ satisfying $M > m \geq m_0$ and $0 \leq k \leq M + 1 - m$, set

$$h_M(k) = \sup_{x:|x| \geq \rho^{m+k}-1} \mathrm{P}_{x,\omega}\left(T_{V_{\rho^m}} < \tau_{V_{\rho^M}}\right).$$

Then h_M solves the difference inequality

$$h_M(k) \leq \frac{2}{5} h_M(k-1) + \frac{3}{5} h_M(k+1)$$

with boundary conditions $h_M(0) = 1$, $h_M(M+1-m) = 0$. Further, by either applying a discrete maximum principle or by a direct computation, we see that $h_M \leq \overline{h}_M$, where $\overline{h}_M$ is the solution of the difference equality

$$\overline{h}_M(k) = \frac{2}{5}\overline{h}_M(k-1) + \frac{3}{5}\overline{h}_M(k+1) \tag{50}$$

with boundary conditions $\overline{h}_M(0) = 1$, $\overline{h}_M(M+1-m) = 0$. Solving (50), we get

$$\overline{h}_M(k) = \frac{1}{1-(3/2)^{M+1-m}} + \frac{1}{1-(2/3)^{M+1-m}}\left(\frac{2}{3}\right)^k.$$

Letting $M \to \infty$, we deduce that for $|x| \geq \rho^{m+k}$,

$$\mathrm{P}_{x,\omega}\left(T_{V_{\rho^m}} < \infty\right) \leq \lim_{M\to\infty} \overline{h}_M(k) = \left(\frac{2}{3}\right)^k. \tag{51}$$

Together with (46), this proves that for almost all $\omega \in \Omega$, the random walk is transient under $\mathrm{P}_{\cdot,\omega}$. $\square$

8 Mean sojourn times in the ball

Using the results about the variational difference of the exit measures and the estimates of Section 4, we provide in this section the basis for the proof of Proposition 1.2, which then leads to Theorem 1.3. Recall that we work under Assumption **A2**.

8.1 Preliminaries

Given three real numbers $a \leq b$ and R, we write $[a,b] \cdot R$ for the interval $[aR, bR]$. Recall the definition of h_L and the corresponding coarse graining scheme on V_L from Section 2.1. In this part, we take a closer look at movements in balls $V_t(x)$ inside V_L, where $t > 0$ is large. As in Section 2.1, we let

$$s_t = \frac{t}{(\log t)^3} \quad \text{and} \quad r_t = \frac{t}{(\log t)^{15}}.$$

We transfer the coarse graining schemes on V_L in the obvious way to $V_t(x)$. We write $\hat{\Pi}_t^x$ for the transition probabilities in $V_t(x)$ belonging to $((h_t^x(y))_{y \in V_t(x)}, p_\omega)$, where $h_t^x(\cdot)$ stands for $h_{t,r_t}(\cdot - x)$, which is defined in (6). The kernel $\hat{\pi}_t^x$ is defined similarly, with p_ω replaced by p^{RW}.

For the corresponding Green's functions we use the expressions $\hat{G}_t^x$ and $\hat{g}_t^x$. If we do not keep x as an index, we always mean $x = 0$ as before. Notice that for $y, z \in V_t(x)$, we have $\hat{\pi}_t^x(y,z) = \hat{\pi}_t(y-x, z-x)$ and $\hat{g}_t^x(y,z) = \hat{g}_t(y-x, z-x)$. Plainly, this is in general not true for $\hat{\Pi}_t^x$ and $\hat{G}_t^x$.

We will readily use the fact that for simple random walk starting in $y \in V_L(x)$ (cf. [24], Proposition 6.2.6),

$$L^2 - |y|^2 \leq \mathrm{E}_y\left[\tau_{V_L(x)}\right] \leq (L+1)^2 - |y|^2. \tag{52}$$

Define the "coarse grained" RWRE sojourn times

$$\Lambda_L(x) = 1_{V_L}(x)\, \frac{1}{h_L(x)} \int_{\mathbb{R}_+} \varphi\left(\frac{t}{h_L(x)}\right) \mathrm{E}_{x,\omega}\left[\tau_{V_t(x) \cap V_L}\right] \mathrm{d}t,$$

and the analog for simple random walk,

$$\lambda_L(x) = 1_{V_L}(x)\, \frac{1}{h_L(x)} \int_{\mathbb{R}_+} \varphi\left(\frac{t}{h_L(x)}\right) \mathrm{E}_x\left[\tau_{V_t(x) \cap V_L}\right] \mathrm{d}t.$$

We will also consider the corresponding quantities Λ_t^x, λ_t^x for balls $V_t(x)$. For example,

$$\Lambda_t^x(y) = 1_{V_t(x)}(y)\, \frac{1}{h_t^x(y)} \int_{\mathbb{R}_+} \varphi\left(\frac{s}{h_t^x(y)}\right) \mathrm{E}_{y,\omega}\left[\tau_{V_s(y) \cap V_t(x)}\right] \mathrm{d}s.$$

We often let kernels operate on mean sojourn times from the left. As an example,

$$\hat{G}_{L,r}\Lambda_L(x) = \sum_{y \in V_L} \hat{G}_{L,r}(x,y)\Lambda_L(y).$$

The basis for our inductive scheme is established by

Lemma 8.1. *For environments $\omega \in \mathcal{P}_\varepsilon$, $x \in \mathbb{Z}^d$,*

$$\mathrm{E}_{x,\omega}\left[\tau_L\right] = \hat{G}_{L,r}\Lambda_L(x).$$

In particular,

$$\mathrm{E}_{x}\left[\tau_L\right] = \hat{g}_{L,r}\lambda_L(x).$$

Proof: We take a probability space $(\Xi, \mathcal{A}, \mathbb{Q})$ carrying independently for each $x \in V_L$ a family of independent real-valued random variables $(\xi_x^{(n)})_{n\in\mathbb{N}}$, distributed according to $\frac{1}{h_L(x)}\varphi\left(\frac{t}{h_L(x)}\right)\mathrm{d}t$. For the sake of convenience set $\xi_x^{(n)} = 1$ for all $x \in \mathbb{Z}^d \backslash V_L$ and all $n \in \mathbb{N}$. Define the filtration $\mathcal{G}_n = \sigma\left(X_0, \ldots, X_n, \xi_{X_0}^{(0)}, \ldots, \xi_{X_{n-1}}^{(n-1)}\right)$. Here, X_n is the projection on the nth component of the first factor of $\left(\mathbb{Z}^d\right)^{\mathbb{N}} \times \Xi$. Then $(X_n, \mathcal{G}_n)$ is a Markov chain on $\left(\left(\mathbb{Z}^d\right)^{\mathbb{N}} \times \Xi, \mathcal{G} \otimes \mathcal{A}, \mathrm{P}_{x,\omega} \otimes \mathbb{Q}\right)$ with transition kernel p_ω and starting point x. With $T_0 = 0$, and iteratively

$$T_{n+1} = \inf\left\{m > T_n : X_m \notin V_{\xi_{X_{T_n}}^{(T_n)}}\left(X_{T_n}\right)\right\} \wedge \tau_L,$$

one shows by induction that T_n is a stopping time with respect to $\mathcal{G}_k$. Moreover, in V_L, the coarse grained chain running with transition kernel $\hat{\Pi}(\omega)$ can be obtained from X_n by looking at times T_n, that is by considering $(X_{T_n})_{n\geq 0}$. Denote by $\widetilde{\mathrm{E}}_{x,\omega}$ the expectation with respect to $\widetilde{\mathrm{P}}_{x,\omega} = \mathrm{P}_{x,\omega} \otimes \mathbb{Q}$. Then, using the strong Markov property in the next to last equality,

$$\begin{aligned}
\mathrm{E}_{x,\omega}\left[\tau_L\right] &= \sum_{z\in V_L} \mathrm{E}_{x,\omega}\left[\sum_{n=0}^{\infty} 1_{\{z\}}(X_n) 1_{\{n<\tau_L\}}\right] = \sum_{z\in V_L} \widetilde{\mathrm{E}}_{x,\omega}\left[\sum_{n=0}^{\infty} 1_{\{z\}}(X_n) 1_{\{n<\tau_L\}}\right] \\
&= \sum_{z\in V_L} \widetilde{\mathrm{E}}_{x,\omega}\left[\sum_{n=0}^{\infty} \sum_{k=T_n}^{T_{n+1}-1} 1_{\{z\}}(X_k)\right] \\
&= \sum_{z\in V_L} \widetilde{\mathrm{E}}_{x,\omega}\left[\sum_{n=0}^{\infty} \left(\sum_{y\in V_L} 1_{\{y\}}\left(X_{T_n}\right)\right) \sum_{k=T_n}^{T_{n+1}-1} 1_{\{z\}}(X_k)\right] \\
&= \sum_{y\in V_L} \sum_{n=0}^{\infty} \widetilde{\mathrm{E}}_{x,\omega}\left[1_{\{y\}}\left(X_{T_n}\right) \widetilde{\mathrm{E}}_{x,\omega}\left[\sum_{z\in V_L} \sum_{k=T_n}^{T_{n+1}-1} 1_{\{z\}}\left(X_k\right) \;\middle|\; \mathcal{G}_{T_n}\right]\right] \\
&= \sum_{y\in V_L} \sum_{n=0}^{\infty} \widetilde{\mathrm{E}}_{x,\omega}\left[1_{\{y\}}\left(X_{T_n}\right)\right] \Lambda_L(y) = \hat{G}_{L,r}\Lambda_L(x).
\end{aligned}$$

□

Note that the proof of the statement does not depend on the particular form of the coarse graining scheme.

8.2 Good and bad points

As in our study of exit laws, we introduce the terminology of good and bad points, but now with respect to both space and time. It turns out that we need simultaneous control over two levels, which is reflected in a stronger notion of "goodness".

Space-good and space-bad points

We say that $x \in V_L$ is *space-good*, if

- $x \in V_L \backslash \mathcal{B}_L$, that is x is good in the sense of Section 2.2.
- If $\mathrm{d}_L(x) > 2s_L$, then additionally for all $t \in [h_L(x), 2h_L(x)]$ and for all $y \in V_t(x)$,
  - For all $t' \in [h_t^x(y), 2h_t^x(y)]$, $||(\Pi_{V_{t'}(y)} - \pi_{V_{t'}(y)})(y,\cdot)||_1 \le \delta$.
  - If $t - |y-x| > 2r_t$, then additionally

$$\left|\left|(\hat{\Pi}_t^x - \hat{\pi}_t^x)\hat{\pi}_t^x(y,\cdot)\right|\right|_1 \le (\log h_t^x(y))^{-9}.$$

A point $x \in V_L$ which is not space-good is called *space-bad*. The set of all space-bad points inside V_L is denoted by $\mathcal{B}_L^{\mathrm{sp}}$. We classify the environments into $\mathrm{Good}_L^{\mathrm{sp}} = \{\mathcal{B}_L^{\mathrm{sp}} = \emptyset\}$ and $\mathrm{Bad}_L^{\mathrm{sp}} = \{\mathcal{B}_L^{\mathrm{sp}} \neq \emptyset\}$. Notice that $\mathcal{B}_L \subset \mathcal{B}_L^{\mathrm{sp}}$ and $\mathrm{Good}_L^{\mathrm{sp}} \subset \mathrm{Good}_L$. As an immediate consequence of the definition,

Lemma 8.2. *There exists $C > 0$ such that if $\delta > 0$ is small, then on $\mathrm{Good}_L^{\mathrm{sp}}$,*

(i) $\hat{G}_{L,r_L} \preceq C\Gamma_{L,r_L}$.

(ii) *If $x \in V_L$ with $\mathrm{d}_L(x) > 2s_L$, then for all $t \in [h_L(x), 2h_L(x)]$,*

$$\hat{G}_t^x \preceq C\Gamma_{t,r_t}(\cdot - x, \cdot - x).$$

Proof: (i) Since $\mathrm{Good}_L^{\mathrm{sp}} \subset \mathrm{Good}_L$, we have $\hat{G} = \hat{G}^g$ on $\mathrm{Good}_L^{\mathrm{sp}}$, and Lemma 4.1 can be applied.
(ii) Take x and t as in the statement. On $\mathrm{Good}_L^{\mathrm{sp}}$, the kernel $\hat{G}_t^x$ coincides with its goodified version, since within $V_t(x)$, there are no bad points. The claim now follows again from Lemma 4.1. □

Lemma 8.3. *If L_1 is large enough, then* $\mathbf{C1}(\delta, L_1)$ *implies that for $L_1 \le L \le L_1(\log L_1)^2$,*

$$\mathbb{P}(\mathrm{Bad}_L^{\mathrm{sp}}) \le \exp\left(-(2/3)(\log L)^2\right).$$

Proof: One can proceed as in the proof of Lemma 2.1. We omit the details. □

Time-good and time-bad points

We will also judge points inside V_L according to their influence on the time the RWRE spends in the ball. Remember the definitions of f_η and condition $\mathbf{C2}(\eta, L_1)$ from Section 1.3. We fix $0 < \eta < 1$. For points in the bulk, we again shall control two levels. We say that a point $x \in V_L$ is *time-good* if the following holds:

- For all $x \in V_L$, $t \in [h_L(x), 2h_L(x)]$,

$$\mathrm{E}_{x,\omega}\left[\tau_{V_t(x)}\right] \in [1 - f_\eta(s_L),\, 1 + f_\eta(s_L)] \cdot \mathrm{E}_x\left[\tau_{V_t(x)}\right]$$

- If $\mathrm{d}_L(x) > 2s_L$, then additionally for all $t \in [h_L(x), 2h_L(x)]$, $y \in V_t(x)$ and for all $t' \in [h_t^x(y), 2h_t^x(y)]$,

$$\mathrm{E}_{y,\omega}\left[\tau_{V_{t'}(y)}\right] \in [1 - f_\eta(s_t),\, 1 + f_\eta(s_t)] \cdot \mathrm{E}_y\left[\tau_{V_{t'}(y)}\right].$$

A point $x \in V_L$ which is not time-good is called *time-bad.* We denote by $\mathcal{B}_L^{\mathrm{tm}} = \mathcal{B}_L^{\mathrm{tm}}(\omega)$ the set of all time-bad points inside V_L. Recall the definition $\mathcal{D}_L$ from Section 2. We let $\mathrm{OneBad}_L^{\mathrm{tm}} = \{\mathcal{B}_L^{\mathrm{tm}} \subset D \text{ for some } D \in \mathcal{D}_L\}$, $\mathrm{ManyBad}_L^{\mathrm{tm}} = (\mathrm{OneBad}_L^{\mathrm{tm}})^c$, and $\mathrm{Good}_L^{\mathrm{tm}} = \{\mathcal{B}_L^{\mathrm{tm}} = \emptyset\} \subset \mathrm{OneBad}_L^{\mathrm{tm}}$.

Important remark

The second point in the definition of time-good provides control over coarse grained mean times on the preceding level, which will be crucial for the proof of Lemma 8.6. Let us look at the first point. If $x \in V_L$ is time-good and $\mathrm{d}_L(x) > r_L$, then by definition of the coarse-graining,

$$\Lambda_L(x) \in [1 - f_\eta(s_L),\, 1 + f_\eta(s_L)] \cdot \lambda_L(x).$$

If $x \in V_L$ is time-good and $\mathrm{d}_L(x) \leq r_L$, then at least

$$\Lambda_L(x) \leq (1 + f_\eta(s_L))\, \mathrm{E}_x\left[\tau_{V_{r_L}(x)}\right].$$

Due to (52), this implies

$$\Lambda_L(x) \leq C(\log L)^{-6} L^2 \quad \text{for all time-good } x \in V_L.$$

However, time-bad points could possibly be very bad and give rise to a sojourn time which is visible on many subsequent larger scales. For example, assume that all transition probabilities inside a ball of radius L have the tendency to push the walker towards the center of the ball (see Figure 7). Then the mean sojourn time will be of order $\exp(cL)$ for some $c > 0$. The probability of such an event should however be exponentially small in the volume L^d. Of course, between this extreme case and a well-behaved environment, there are many intermediate configurations. One needs to show that "very (time-)bad" environments do not occur too often, which seems to be a challenging problem. This is the point where Assumption **A2** helps out. It allows us to concentrate on the event

$$\mathrm{NotTooBad}_L^{\mathrm{tm}} = \left\{\omega \in \Omega : \Lambda_L(x) \leq (\log L)^{-2} L^2 \text{ for all } x \in V_L\right\}. \tag{53}$$

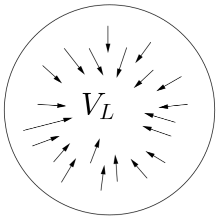

Figure 7: A “trap”: Starting at the origin, the walker is pushed back to the center, no matter in which direction he walks. On average, he needs time of order $\exp(cL)$ to leave the ball.

Lemma 8.4. *If* **A2** *holds, then for L sufficiently large,*

$$\mathbb{P}\left((\mathrm{NotTooBad}_L^{\mathrm{tm}})^c\right) \leq (1/2)L^{-6d}.$$

Proof: First notice that with

$$E_L = \left\{\omega \in \Omega : \text{for all } x \in V_L, \text{with } t = 2h_L(x),\ \mathrm{E}_{x,\omega}\left[\tau_{V_t(x)}\right] \leq (\log L)^{-2}L^2\right\}$$

we have $E_L \subset \mathrm{NotTooBad}_L^{\mathrm{tm}}$. As $2h_L(x) \leq s_L/10 < (\log L)^{-3}L$, the complement of E_L is bounded under **A2** by $\mathbb{P}(E_L^c) \leq CL^dL^{-8d}$.

□

Remark 8.1. (i) On a certain class of environments, we can easily bound the mean time the RWRE spends the a ball. Fix a unit vector $e \in \{e_i\}_{i=1}^d$ from the canonical basis of $\mathbb{Z}^d$. We consider an environment $\omega \in \mathcal{P}_\varepsilon$ such that for each $x \in \mathbb{Z}^d$, $\omega_x(e) = \omega_x(-e)$, i.e. the environment is *balanced* in direction e. In such a case,

$$M_n = (X_n \cdot e)^2 - \sum_{k=0}^{n-1}\left(\omega_{X_k}(e) + \omega_{X_k}(-e)\right)$$

is a $\mathrm{P}_{0,\omega}$-martingale with respect to the filtration generated by the walk $(X_n)_{n\geq 0}$. By the stopping theorem, $\mathrm{E}_{0,\omega}\left[M_{n\wedge\tau_L}\right] = 0$. Since $\omega_{X_k}(e) + \omega_{X_k}(-e) \geq 1/d - 2\varepsilon > 0$, it follows that

$$\mathrm{E}_{0,\omega}\left[n \wedge \tau_L\right] \leq (1/d - 2\varepsilon)^{-1}\,\mathrm{E}_{0,\omega}\left[(X_{n\wedge\tau_L} \cdot e)^2\right].$$

Therefore,

$$\mathrm{E}_{0,\omega}\left[\tau_L\right] \leq \frac{d}{1 - 2\varepsilon d}(L+1)^2,$$

and **A2** is trivially satisfied.

However, for measures μ which are invariant under rotations, the class of such environments has positive measure under $\mathbb{P}_\mu$ only if μ is supported on the subset of symmetric transition probabilities $\{q \in \mathcal{P}_\varepsilon : q(+e_i) = q(-e_i) \text{ for all } i = 1, \ldots, d\}$, implying $\omega_x(e) = \omega_x(-e)$ for all unit vectors e and $x \in \mathbb{Z}^d$ almost surely. In this case, $|X_n|^2 - n$ is a quenched martingale, and $L^2 \leq \mathrm{E}_{0,\omega}\left[\tau_L\right] \leq (L+1)^2$ for almost all environments.
(ii) Before proceeding, let us mention that Assumption **A2** can be expressed in terms of hitting probabilities. For example, if there exists $\rho > 0$ such that for L large,

$$\mathbb{P}\left(\inf_{x\in V_L} \mathrm{P}_{x,\omega}\left(\tau_L \leq (\log L)^3L^2\right) \geq \rho\right) \geq 1 - L^{-8d},$$

then **A2** holds. Indeed, on the event $\{\inf_{x\in V_L} \mathrm{P}_{x,\omega}(\tau_L \le (\log L)^3 L^2) \ge \rho\}$,

$$\mathrm{E}_{0,\omega}[\tau_L] \le (\log L)^3 L^2 + \sum_{k=1}^{\infty} \mathrm{P}_{0,\omega}\left(\tau_L > k(\log L)^3 L^2\right)(\log L)^3 L^2.$$

By the Markov property it follows that on this event,

$$\mathrm{P}_{0,\omega}\left(\tau_L > k(\log L)^3 L^2\right) \le (1-\rho)^k,$$

whence for large L,

$$\mathrm{E}_{0,\omega}[\tau_L] \le (1/\rho)(\log L)^3 L^2 \le (\log L)^4 L^2.$$

Let us continue by showing that we can forget about environments with space-bad points or widely spread time-bad points.

Lemma 8.5. *If L_1 is large, then* $\mathbf{C1}(\delta, L_1)$, $\mathbf{C2}(\eta, L_1)$ *imply that for L with $L_1 \le L \le L_1(\log L_1)^2$,*

$$\mathbb{P}\left(\mathrm{Bad}_L^{\mathrm{sp}} \cup \mathrm{ManyBad}_L^{\mathrm{tm}}\right) \le (1/2)L^{-6d}.$$

Proof: We have $\mathbb{P}\left(\mathrm{Bad}_L^{\mathrm{sp}} \cup \mathrm{ManyBad}_L^{\mathrm{tm}}\right) \le \mathbb{P}\left(\mathrm{Bad}_L^{\mathrm{sp}}\right) + \mathbb{P}\left(\mathrm{ManyBad}_L^{\mathrm{tm}}\right)$. The first summand is bounded by Lemma 8.3. For the second, it follows from the definition of time-badness, f and (52) that if $x \in \mathcal{B}_L^{\mathrm{tm}}$ and L is large, then either

$$\mathrm{E}_{x,\omega}\left[\tau_{V_t(x)}\right] \notin [1-f_\eta(t),\, 1+f_\eta(t)] \cdot \mathrm{E}_x\left[\tau_{V_t(x)}\right]$$

for some $t \in [h_L(x), 2h_L(x)] \cap \mathbb{N}$, or, if $\mathrm{d}_L(x) > 2s_L$,

$$\mathrm{E}_{y,\omega}\left[\tau_{V_{t'}(y)}\right] \notin [1-f_\eta(t'),\, 1+f_\eta(t')] \cdot \mathrm{E}_y\left[\tau_{V_{t'}(y)}\right]$$

for some $y \in V_{2h_L(x)}(x)$, $t' \in [h^x_{h_L(x)}(y), 2h^x_{2h_L(x)}(y)] \cap \mathbb{N}$.

Now notice that for all $x \in V_L$, we have $h_L(x) \ge r_L/20$. Moreover, if $\mathrm{d}_L(x) > 2s_L$, then $h_L(x) = s_L/20$, whence for all $y \in V_t(x)$, $t \in [h_L(x), 2h_L(x)]$, it follows that $h^x_t(y) \ge r_{(s_L/20)}/20$. We conclude that under $\mathbf{C2}(\eta, L_1)$,

$$\mathbb{P}\left(x \in \mathcal{B}_L^{\mathrm{tm}}\right) \le s_L\left(r_L/20\right)^{-6d} + CL^d s_{s_L}\left(r_{s_L/20}/20\right)^{-6d},$$

and therefore

$$\mathbb{P}\left(\mathrm{ManyBad}_L^{\mathrm{tm}}\right) \le CL^{4d+2}\left(r_{s_L/20}/20\right)^{-12d} \le (1/3)L^{-6d}.$$

□

8.3 Estimates on mean times

It remains to deal with environments $\omega \in \mathrm{Good}_L^{\mathrm{sp}} \cap \mathrm{OneBad}_L^{\mathrm{tm}} \cap \mathrm{NotTooBad}_L^{\mathrm{tm}}$. In contrast to the estimates on exit measures, we treat all these environments at once. The main statement of this section, Lemma 8.8, can therefore be seen as the analog for sojourn times of both Lemmata 5.2 and 5.3 . In the following, we will always assume that δ and L are such that Lemma 8.2 can be applied. We start with two auxiliary statements. Here, the difference estimates on the coarse grained Green's functions from Section 4.3 play a crucial role.

Lemma 8.6. *Let $0 \leq \alpha < 3$ and $x, y \in V_{L-2s_L} \backslash \mathcal{B}_L^{\mathrm{tm}}$ with $|x-y| \leq (\log s_L)^{-\alpha} s_L$. On $\mathrm{Good}_L^{\mathrm{sp}}$,*

$$|\Lambda_L(x) - \Lambda_L(y)| \leq C(\log\log s_L)(\log s_L)^{-\alpha} s_L^2.$$

Proof: The claim follows if we show that for all $t \in [(1/20)s_L, (1/10)s_L]$,

$$\left|\mathrm{E}_{x,\omega}\left[\tau_{V_t(x)}\right] - \mathrm{E}_{y,\omega}\left[\tau_{V_t(y)}\right]\right| \leq C(\log\log t)(\log t)^{-\alpha} t^2.$$

Set $t' = (1 - 20(\log t)^{-\alpha})\, t$. Then $V_{t'}(x) \subset V_t(x) \cap V_t(y)$. Further, let $B = V_{t'-2s_t}(x)$. By Lemma 8.1,

$$\mathrm{E}_{x,\omega}\left[\tau_{V_t(x)}\right] = \hat{G}_t^x 1_B \Lambda_t^x(x) + \hat{G}_t^x 1_{V_t(x)\backslash B} \Lambda_t^x(x). \tag{54}$$

Since $x \in V_{L-2s_L} \backslash \mathcal{B}_L^{\mathrm{tm}}$, it follows that $\Lambda_t^x(z) \leq C(\log t)^{-6} t^2$, for all $z \in V_t(x)$. Moreover, since $\omega \in \mathrm{Good}_L^{\mathrm{sp}}$, we have by Lemma 8.2 $\hat{G}_t^x \preceq C\Gamma_{t,r_t}(\cdot - x, \cdot - x)$. Thus, Lemma 4.3 (iv) yields

$$\hat{G}_t^x 1_{V_t(x)\backslash B} \Lambda_t^x(x) \leq C\Gamma_{t,r_t}\left(0, V_t \backslash V_{t'-2s_t}\right) (\log t)^{-6} t^2 \leq (\log t)^{-\alpha} t^2.$$

for L (and therefore also t) sufficiently large. Concerning $\mathrm{E}_{y,\omega}\left[\tau_{V_t(y)}\right]$, we split again into

$$\mathrm{E}_{y,\omega}\left[\tau_{V_t(y)}\right] = \hat{G}_t^y 1_B \Lambda_t^y(y) + \hat{G}_t^y 1_{V_t(y)\backslash B} \Lambda_t^y(y).$$

As above, the second summand is bounded by $(\log t)^{-\alpha} t^2$. For $z \in B$, we have $h_t^x(z) = h_t^y(z) = (1/20)s_t$. In particular, $\hat{\Pi}_t^x(z, \cdot) = \hat{\Pi}_t^y(z, \cdot)$, and also $\Lambda_t^x(z) = \Lambda_t^y(z)$. Since both x and y are contained in $B \subset V_t(x) \cap V_t(y)$, the strong Markov property gives

$$\hat{G}_t^y(y, z) = \hat{G}_t^x(y, z) + b(y, z),$$

where

$$b(y, z) = \mathrm{E}_{y,\hat{\Pi}_t^y(\omega)}\left[\hat{G}_t^y(\tau_B, z);\, \tau_B < \infty\right] - \mathrm{E}_{y,\hat{\Pi}_t^x(\omega)}\left[\hat{G}_t^x(\tau_B, z);\, \tau_B < \infty\right].$$

Therefore,

$$\begin{aligned}
&\left|\mathrm{E}_{x,\omega}\left[\tau_{V_t(x)}\right] - \mathrm{E}_{y,\omega}\left[\tau_{V_t(y)}\right]\right| \\
&\quad \leq \; 2(\log t)^{-\alpha} t^2 + \sum_{z \in B}\left(\left|\hat{G}_t^x(x, z) - \hat{G}_t^x(y, z)\right| + |b(y, z)|\right) \Lambda_t^x(z).
\end{aligned}$$

The quantity $\Lambda_t^x(z)$ is estimated as above. For the sum over $|b(y,z)|$, we notice that if $w \in V_t(y) \backslash B$, then $t - |w-y| \le C(\log t)^{-\alpha} t$. We can use twice Lemma 4.3 (v) to get

$$\sum_{z \in B} |b(y,z)| \le \sup_{v \in V_t(x) \backslash B} \hat{G}_t^x(v,B) + \sup_{w \in V_t(y) \backslash B} \hat{G}_t^y(w,B) \le C(\log t)^{6-\alpha}.$$

Finally, for the sum over the Green's function difference, we recall that $\hat{G}_t^x$ coincides with its goodified version, so we may apply Lemma 4.4. Doing so $O\left((\log t)^{3-\alpha}\right)$ times gives

$$\sum_{z \in B} \left| \hat{G}_t^x(x,z) - \hat{G}_t^x(y,z) \right| \le C(\log\log t)(\log t)^{6-\alpha}.$$

This proves the statement. $\square$

Lemma 8.7. *Set* $\Delta = 1_{V_L}(\hat{\Pi}_{L,r_L} - \hat{\pi}_{L,r_L})$. *On* $\mathrm{Good}_L^{\mathrm{sp}} \cap \mathrm{OneBad}_L^{\mathrm{tm}} \cap \mathrm{NotTooBad}_L^{\mathrm{tm}}$,

$$\sup_{x \in V_L} \left| \hat{G}_{L,r_L} \Delta \hat{g}_{L,r_L} \Lambda_L(x) \right| \le C(\log L)^{-5/3} L^2.$$

Proof: We have

$$\hat{G}\Delta\hat{g}\Lambda_L(x) = \hat{G}\Delta\hat{\pi}\hat{g}\Lambda_L(x) + \hat{G}\Delta\Lambda_L(x) = A_1 + A_2.$$

By Lemma 8.2, $\hat{G} = \hat{G}^g \preceq C\Gamma$. Therefore, with $B_1 = V_{L-2r_L}$, we bound A_1 by

$$\begin{aligned}
|A_1| &\le \left| \hat{G} 1_{B_1} \Delta \hat{\pi} \hat{g} \Lambda_L(x) \right| + \left| \hat{G} 1_{B_1^c} \Delta \hat{\pi} \hat{g} \Lambda_L(x) \right| \\
&\le \left| \sum_{v \in B_1, w \in V_L} \hat{G}(x,v) \Delta \hat{\pi}(v,w) \sum_{y \in V_L} \left(\hat{g}(w,y) - \hat{g}(v,y) \right) \Lambda_L(y) \right| + C(\log L)^{-2} L^2 \\
&\le C(\log L)^{-5/3} L^2,
\end{aligned}$$

where in the next to last inequality we have used the bound on $\Lambda_L(y)$ coming from (53), Lemma 4.3 (iv), (v) and in the last additionally Lemma 4.4. For the term A_2, we let

$$U(\mathcal{B}_L^{\mathrm{tm}}) = \{ v \in V_L : |\Delta(v,w)| > 0 \text{ for some } w \in \mathcal{B}_L^{\mathrm{tm}} \}$$

and define $B = V_{L-5s_L} \backslash U(\mathcal{B}_L^{\mathrm{tm}})$. We split into

$$A_2 = \hat{G} 1_B \Delta \Lambda_L(x) + \hat{G} 1_{B^c} \Delta \Lambda_L(x).$$

Lemma 4.3 (iv) and an analogous application of Corollary 4.1 with $U(\mathcal{B}_L^{\mathrm{tm}})$ instead of $\mathcal{B}_L$ yield

$$\hat{G}(x, U(\mathcal{B}_L^{\mathrm{tm}}) \cup \mathrm{Sh}_L(5s_L)) \le C \log\log L.$$

Since $\Lambda_L(y) \le (\log L)^{-2} L^2$, this estimates the second summand of A_2. For the first one,

$$\hat{G} 1_B \Delta \Lambda_L(x) \le C\Gamma(x,B) \sup_{v \in B} |\Delta \Lambda_L(v)|.$$

Since $\Gamma(x,B) \le C(\log L)^6$, the claim follows we show that for $v \in B$,

$$|\Delta\Lambda_L(v)| \le C(\log L)^{-8}L^2, \tag{55}$$

which, by definition of Δ, in turn follows if for all $t \in [h_L(v), 2h_L(v)]$,

$$\left|\left(\Pi_{V_t(v)} - \pi_{V_t(v)}\right)\Lambda_L(v)\right| \le C(\log L)^{-8}L^2.$$

Notice that on B, $h_L(\cdot) = (1/20)s_L$. We now fix $v \in B$ and $t \in [(1/20)s_L, (1/10)s_L]$. Set $\Delta' = 1_{V_t(v)}(\hat{\Pi}_t^v - \hat{\pi}_t^v)$ and $B' = V_{t-2r_t}(v)$. By expansion (3),

$$\left(\Pi_{V_t(v)} - \pi_{V_t(v)}\right)\Lambda_L(v) = \hat{G}_t^v 1_{B'}\Delta'\pi_{V_t(v)}\Lambda_L(v) + \hat{G}_t^v 1_{V_t(v)\backslash B'}\Delta'\pi_{V_t(v)}\Lambda_L(v). \tag{56}$$

Since $\pi_{V_t(v)} = \hat{\pi}_t^v\pi_{V_t(v)}$, we get

$$\begin{aligned}
\left|\hat{G}_t^v 1_{B'}\Delta'\pi_{V_t(v)}\Lambda_L(v)\right| &\le \hat{G}_t^v(v,B')\sup_{w\in B'}||\Delta'\hat{\pi}_t^v(w,\cdot)||_1 \sup_{y\in\partial V_t(v)}\Lambda_L(y)\\
&\le C(\log s_L)^6\sup_{w\in B'}(\log h_t^v(w))^{-9}(\log L)^{-6}L^2\\
&\le C(\log L)^{-9}L^2.
\end{aligned}$$

Here, in the next to last inequality we have used the fact that v is space-good, all $y \in \partial V_t(v)$ are time-good, and Lemma 4.3 (v). The last inequality follows from the bound $h_t^v(w) \ge (1/20)r_{s_L/20}$. For the second summand of (56), Lemma 4.3 (iv) gives $\hat{G}_t^v(v, V_t(v)\backslash B') \le C$, whence

$$\left|\hat{G}_t^v 1_{V_t(v)\backslash B'}\Delta'\pi_{V_t(v)}\Lambda_L(v)\right| \le C\sup_{w\in V_t(v)\backslash B'}\left|\Delta'\pi_{V_t(v)}\Lambda_L(w)\right|.$$

Fix $w \in V_t(v)\backslash B'$. Set $\eta = \mathrm{d}(w, \partial V_t(v)) \le 2r_t + \sqrt{d}$ and choose $y_w \in \partial V_t(v)$ such that $|w - y_w| = \eta$. With

$$I(y_w) = \left\{y \in \partial V_t(v) : |y - y_w| \le (\log L)^{-5/2}s_L\right\},$$

we write

$$\begin{aligned}
&\Delta'\pi_{V_t(v)}\Lambda_L(w)\\
&\quad= \sum_{y\in\partial V_t(v)}\Delta'\pi_{V_t(v)}(w,y)\left(\Lambda_L(y) - \Lambda_L(y_w)\right)\\
&\quad= \sum_{y\in I(y_w)}\Delta'\pi_{V_t(v)}(w,y)\left(\Lambda_L(y) - \Lambda_L(y_w)\right)\\
&\qquad+ \sum_{y\in\partial V_t(v)\backslash I(y_w)}\Delta'\pi_{V_t(v)}(w,y)\left(\Lambda_L(y) - \Lambda_L(y_w)\right).
\end{aligned} \tag{57}$$

For $y \in I(y_w)$, Lemma 8.6 yields $|\Lambda_L(y) - \Lambda_L(y_w)| \le C(\log L)^{-7/3}s_L^2$. Therefore,

$$\sum_{y\in I(y_w)}\left|\Delta'\pi_{V_t(v)}(w,y)\right|\left|\Lambda_L(y) - \Lambda_L(y_w)\right| \le C(\log L)^{-8}L^2.$$

It remains to handle the second term of (57). To this end, let $U(w) = \{u \in V_t(v) : |\Delta'(w,u)| > 0\}$. Using for $y \in \partial V_t(v) \backslash I(y_w)$ the simple bound $|\Lambda_L(y) - \Lambda_L(y_w)| \leq \Lambda_L(y) + \Lambda_L(y_w) \leq C(\log L)^{-6}L^2$,

$$\begin{aligned} &\left| \sum_{y \in \partial V_t(v) \backslash I(y_w)} \Delta' \pi_{V_t(v)}(w,y)\left(\Lambda_L(y) - \Lambda_L(y_w)\right) \right| \\ &\quad \leq \ C(\log L)^{-6} L^2 \sup_{u \in U(w)} \pi_{V_t(v)}\left(u, \partial V_t(v) \backslash I(y_w)\right). \end{aligned}$$

If $u \in U(w)$ and $y \in \partial V_t(v) \backslash I(y_w)$, then

$$|u - y| \geq |y - y_w| - |y_w - u| \geq (\log L)^{-5/2} s_L - 3r_t \geq (1/2)(\log L)^{-5/2} s_L.$$

For such u, we get by Lemma 3.2 (ii) and Lemma 3.4

$$\begin{aligned} \pi_{V_t(v)}\left(u, \partial V_t(v) \backslash I(y_w)\right) &\leq \ C r_t \sum_{y \in \partial V_t(v) \backslash I(y_w)} \frac{1}{|u-y|^d} \leq C r_t (\log L)^{5/2} (s_L)^{-1} \\ &\leq \ C(\log L)^{-9}. \end{aligned}$$

This bounds the second term of (57). We have proved (55) and hence the lemma. □

Now it is easy to prove

Lemma 8.8. *There exists $L_0 = L_0(\eta)$ such that for $L \geq L_0$ and environments $\omega \in \mathrm{Good}_L^{\mathrm{sp}} \cap \mathrm{OneBad}_L^{\mathrm{tm}} \cap \mathrm{NotTooBad}_L^{\mathrm{tm}}$,*

$$\mathrm{E}_{0,\omega}\left[\tau_L\right] \in \left[1 - f_\eta(L),\, 1 + f_\eta(L)\right] \cdot \mathrm{E}_0\left[\tau_L\right].$$

Proof: By Lemma 8.1 and perturbation expansion (3), with $\Delta = 1_{V_L}(\hat{\Pi} - \hat{\pi})$,

$$\mathrm{E}_{0,\omega}\left[\tau_L\right] = \hat{G}\Lambda_L(0) = \hat{g}\Lambda_L(0) + \hat{G}\Delta\hat{g}\Lambda_L(0) = A_1 + A_2.$$

Set $B = V_{L-r_L} \backslash \mathcal{B}_L^{\mathrm{tm}}$. The term A_1 we split into

$$A_1 = \hat{g}1_B\Lambda_L(0) + \hat{g}1_{V_L \backslash B}\Lambda_L(0).$$

Since $\hat{g}(0, V_L \backslash B) \leq C$ and $\Lambda_L(x) \leq (\log L)^{-2}L^2$, the second summand of A_1 can be bounded by $O\left((\log L)^{-2}\right)\mathrm{E}_0[\tau_L]$. The main contribution comes from the first summand. First notice that

$$\hat{g}1_B\lambda_L(0) = \mathrm{E}_0\left[\tau_L\right]\left(1 + O\left((\log L)^{-6}\right)\right).$$

Further, we have for $x \in B$,

$$\Lambda_L(x) \in \left[1 - f_\eta\left((\log L)^{-3}L\right),\, 1 + f_\eta\left((\log L)^{-3}L\right)\right] \cdot \lambda_L(x).$$

Collecting all terms, we conclude that

$$\begin{aligned} A_1 \in &\left[1 - O\left((\log L)^{-2}\right) - f_\eta\left((\log L)^{-3}L\right),\, 1 + O\left((\log L)^{-2}\right) + f_\eta\left((\log L)^{-3}L\right)\right] \\ &\times \mathrm{E}_0\left[\tau_L\right]. \end{aligned}$$

Lemma 8.7 bounds A_2 by $O((\log L)^{-5/3})\,\mathrm{E}_0[\tau_L]$. Since for L sufficiently large,

$$f_\eta(L) > f_\eta\left((\log L)^{-3}L\right) + C(\log L)^{-5/3},$$

we arrive at

$$\mathrm{E}_{0,\omega}\left[\tau_L\right] = A_1 + A_2 \in \left[1 - f_\eta(L),\, 1 + f_\eta(L)\right] \cdot \mathrm{E}_0\left[\tau_L\right].$$

□

9 Proofs of the main results on sojourn times

Proof of Proposition 1.2: (i) From Lemmata 8.4, 8.5 and 8.8 we deduce that for large L_0, if $L_1 \geq L_0$ and $L_1 \leq L \leq L_1(\log L_1)^2$, we have under $\mathbf{C1}(\delta, L_1)$ and $\mathbf{C2}(\eta, L_1)$

$$\begin{aligned}
&\mathbb{P}\left(\mathrm{E}_{0,\omega}\left[\tau_L\right] \notin \left[1 - f(L),\, 1 + f(L)\right] \cdot \mathrm{E}_0\left[\tau_L\right]\right)\\
&\quad\leq\ \mathbb{P}\left(\mathrm{Bad}_L^{\mathrm{sp}} \cup \mathrm{ManyBad}_L^{\mathrm{tm}}\right) + \mathbb{P}\left((\mathrm{NotTooBad}_L^{\mathrm{tm}})^c\right)\\
&\qquad + \mathbb{P}\left(\{\mathrm{E}_{0,\omega}\left[\tau_L\right] \notin \left[1 - f(L),\, 1 + f(L)\right] \cdot \mathrm{E}_0[\tau_L]\}\right.\\
&\qquad\qquad \left.\cap\, \mathrm{Good}_L^{\mathrm{sp}} \cap \mathrm{OneBad}_L^{\mathrm{tm}} \cap \mathrm{NotTooBad}_L^{\mathrm{tm}}\right)\\
&\quad\leq\ L^{-6d}.
\end{aligned}$$

By Proposition 1.1, if $\delta > 0$ is small, $\mathbf{C1}(\delta, L)$ holds under $\mathbf{A0}(\varepsilon)$ for all large L, provided $\varepsilon \leq \varepsilon_0(\delta)$. This proves part (i) of the proposition.
(ii) We take L_0 from part (i). By choosing ε small enough, we can guarantee that $\mathbf{C2}(\eta, L_0)$ holds. Then, by what we just proved, $\mathbf{C2}(\eta, L)$ holds for all $L \geq L_0$. Recalling (52), we therefore have for large $L \geq L_0$

$$\begin{aligned}
&\mathbb{P}\left(\sup_{x:|x|\leq L^3}\ \sup_{y\in V_L(x)} \mathrm{E}_{y,\omega}\left[\tau_{V_L(x)}\right] \notin [1-\eta, 1+\eta] \cdot L^2\right)\\
&\quad\leq\ CL^{3d}\,\mathbb{P}\left(\sup_{y\in V_L} \mathrm{E}_{y,\omega}\left[\tau_L\right] \notin [1-\eta, 1+\eta] \cdot L^2\right)\\
&\quad\leq\ CL^{3d}\,\mathbb{P}\left(\mathrm{E}_{0,\omega}\left[\tau_L\right] < (1-\eta) \cdot L^2\right) + CL^{3d}\,\mathbb{P}\left(\sup_{y\in V_L} \mathrm{E}_{y,\omega}\left[\tau_L\right] > (1+\eta) \cdot L^2\right)\\
&\quad\leq\ CL^{-3d} + CL^{4d}\,\mathbb{P}\left(\mathrm{E}_{0,\omega}\left[\tau_L\right] > (1+\eta) \cdot L^2\right) \leq L^{-d}.
\end{aligned}$$

□

Proof of Corollary 1.2: First let $k = 1$. Using Proposition 1.2 (ii) and Borel-Cantelli, we obtain for $\mathbb{P}$-almost all ω

$$\limsup_{L\to\infty}\ \sup_{x:|x|\leq L^3}\ \sup_{y\in V_L(x)} \mathrm{E}_{y,\omega}\left[\tau_{V_L(x)}\right]/L^2 \leq 2. \tag{58}$$

For the rest of the proof, take an environment ω satisfying (58). Assume $k \geq 2$. Then

$$\begin{aligned}
\mathrm{E}_{y,\omega}\left[\tau_{V_L(x)}^k\right] &= \sum_{l_1,\ldots,l_k\geq 0} \mathrm{P}_{y,\omega}\left(\tau_{V_L(x)} > l_1, \ldots, \tau_{V_L(x)} > l_k\right)\\
&\leq\ k! \sum_{0\leq l_1\leq\ldots\leq l_k} \mathrm{P}_{y,\omega}\left(\tau_{V_L(x)} > l_k\right).
\end{aligned}$$

By the Markov property, using the case $k=1$ and induction in the last step,

$$\begin{aligned}
&\sum_{0\leq l_1\leq \ldots\leq l_k} \mathrm{P}_{y,\omega}\left(\tau_{V_L(x)} > l_k\right)\\
&= \sum_{0\leq l_1\leq \ldots\leq l_{k-1}} \mathrm{E}_{y,\omega}\left[\sum_{l=0}^{\infty} \mathrm{P}_{X_{l_{k-1}},\omega}\left(\tau_{V_L(x)} > l\right); \tau_{V_L(x)} > l_{k-1}\right]\\
&\leq \sup_{z\in V_L(x)} \mathrm{E}_{z,\omega}\left[\tau_{V_L(x)}\right] \sum_{0\leq l_1\leq \ldots\leq l_{k-1}} \mathrm{E}_{y,\omega}\left[\tau_{V_L(x)} > l_{k-1}\right]\\
&\leq 2^k L^{2k},
\end{aligned}$$

if $L = L(\omega)$ is sufficiently large. $\square$

Proof of Theorem 1.3: Both statements are proved in the same way, so we restrict ourselves to the lim sup. Set $\overline{\tau}_{V_L(x)} = \tau_{V_L(x)}/L^2$ and

$$B_1 = \left\{\limsup_{L\to\infty} \sup_{x:|x|\leq L^3} \sup_{y\in V_L(x)} \mathrm{E}_{y,\omega}\left[\overline{\tau}_{V_L(x)}\right] \in [1-\eta, 1+\eta]\right\}.$$

By Proposition 1.2 and Borel-Cantelli it follows that $\mathbb{P}(B_1) = 1$ if $\varepsilon \leq \varepsilon_0$. Moreover, on B_1 the conclusion of Corollary 1.2 holds true. Corollary 1.1 tells us that for small enough ε, on a set B_2 of full measure the RWRE satisfies (2) and is therefore transient. Let $B = B_1 \cap B_2$ and define

$$\xi = \begin{cases} \limsup_{L\to\infty} \sup_{x:|x|\leq L^3} \sup_{y\in V_L(x)} \mathrm{E}_{y,\omega}\left[\overline{\tau}_{V_L(x)}\right] & \text{for } \omega\in B\\ 0 & \text{for } \omega \in \Omega\backslash B \end{cases}.$$

Choose an bijective enumeration function $g:\mathbb{Z}^d\to\mathbb{N}$ with $g(0)=0$ and $g(x)<g(y)$ whenever $|x|<|y|$. Let $\mathcal{N}$ denote the collection of all $\mathbb{P}$-null sets in $\mathcal{F}$ and set $\mathcal{F}_n' = \sigma(\mathcal{N}, Z_n, Z_{n+1},\ldots)$, where $Z_k:\Omega\to\mathcal{P}$, $Z_k(\omega) = \omega_{g^{-1}(k)}$, is the projection on the $g^{-1}(k)$-th component. Let $\mathcal{T} = \cap_n \mathcal{F}_n'$ be the (completed) tail σ-field. We show that ξ is measurable with respect to $\mathcal{T}$, implying that ξ is $\mathbb{P}$-almost surely constant. Take $\omega\in B$. We claim that for each fixed ball V_l around the origin, T_l its hitting time,

$$\xi(\omega) = \underbrace{\limsup_{L\to\infty} \sup_{x:|x|\leq L^3} \sup_{y\in V_L(x)\backslash V_l} \mathrm{E}_{y,\omega}\left[\overline{\tau}_{V_L(x)}; T_l=\infty\right]}_{=\xi_l(\omega)}. \tag{59}$$

But then also

$$\xi(\omega) = \lim_{l\to\infty} \xi_l(\omega).$$

Since ξ_l depends only on the random variables ω_x with $|x|>l$, ξ is in fact measurable with respect to $\mathcal{T}$, provided the above representation holds true. Therefore, we only have to prove (59). Obviously, $\xi(\omega)\geq\xi_l(\omega)$. For the other direction, by the Markov property in the first inequality,

$$\begin{aligned}
\mathrm{E}_{y,\omega}\left[\overline{\tau}_{V_L(x)}\right] &= \mathrm{E}_{y,\omega}\left[\overline{\tau}_{V_L(x)};\, \tau_{V_L(x)} \le L\right] + \mathrm{E}_{y,\omega}\left[\overline{\tau}_{V_L(x)};\, \tau_{V_L(x)} > L\right] \\
&\le 2L^{-1} + \mathrm{E}_{y,\omega}\left[\mathrm{E}_{X_L,\omega}\left[\overline{\tau}_{V_L(x)}\right];\, \tau_{V_L(x)} > L\right] \\
&\le 2L^{-1} + \mathrm{E}_{y,\omega}\left[\mathrm{E}_{X_L,\omega}\left[\overline{\tau}_{V_L(x)};\, T_l < \infty\right];\, \tau_{V_L(x)} > L\right] \\
&\quad + \mathrm{E}_{y,\omega}\left[\mathrm{E}_{X_L,\omega}\left[\overline{\tau}_{V_L(x)};\, T_l = \infty\right];\, \tau_{V_L(x)} > L\right].
\end{aligned}$$

Clearly,

$$\mathrm{E}_{y,\omega}\left[\mathrm{E}_{X_L,\omega}\left[\overline{\tau}_{V_L(x)};\, T_l = \infty\right];\, \tau_{V_L(x)} > L\right] \le \sup_{y\in V_L(x)\setminus V_l} \mathrm{E}_{y,\omega}\left[\overline{\tau}_{V_L(x)};\, T_l = \infty\right],$$

so $\xi(\omega) \le \xi_l(\omega)$ will follow if we show that

$$\limsup_{L\to\infty} \sup_{x:|x|\le L^3} \sup_{y\in V_L(x)} \mathrm{E}_{y,\omega}\left[\mathrm{E}_{X_L,\omega}\left[\overline{\tau}_{V_L(x)};\, T_l < \infty\right];\, \tau_{V_L(x)} > L\right] = 0. \tag{60}$$

By Cauchy-Schwarz in the first and Corollary 1.2 in the last inequality, for large L,

$$\begin{aligned}
&\mathrm{E}_{y,\omega}\left[\mathrm{E}_{X_L,\omega}\left[\overline{\tau}_{V_L(x)};\, T_l < \infty\right];\, \tau_{V_L(x)} > L\right] \\
&\le \mathrm{E}_{y,\omega}\left[\mathrm{E}_{X_L,\omega}\left[\overline{\tau}^2_{V_L(x)}\right]^{1/2} \mathrm{P}_{X_L,\omega}\left(T_l < \infty\right)^{1/2};\, \tau_{V_L(x)} > L\right] \\
&\le \sup_{z\in V_L(x)} \mathrm{E}_{z,\omega}\left[\overline{\tau}^2_{V_L(x)}\right]^{1/2} \mathrm{E}_{y,\omega}\left[\mathrm{P}_{X_L,\omega}\left(T_l < \infty\right)^{1/2}\right] \\
&\le 3\,\mathrm{E}_{y,\omega}\left[\mathrm{P}_{X_L,\omega}\left(T_l < \infty\right)^{1/2}\right].
\end{aligned}$$

For the probability inside the expectation, note that as a consequence of (2), for each $\vartheta > 0$ we can choose $K = K(\omega, l)$ such that

$$\sup_{z:|z|\ge K} \mathrm{P}_{z,\omega}\left(T_l < \infty\right) \le \vartheta^2/81.$$

Therefore, replacing the probability by 1 on $\{|X_L| < K\}$,

$$\mathrm{E}_{y,\omega}\left[\mathrm{P}_{X_L,\omega}\left(T_l < \infty\right)^{1/2}\right] \le \vartheta/9 + \mathrm{P}_{y,\omega}\left(|X_L| < K\right).$$

Using again (2), there exists $K' = K'(\omega, K)$ such that

$$\sup_{y:|y|\ge K'} \mathrm{P}_{y,\omega}\left(|X_L| < K\right) \le \sup_{y:|y|\ge K'} \mathrm{P}_{y,\omega}\left(T_K < \infty\right) \le \vartheta/9.$$

On the other hand, for each fixed $K' > 0$ and $K > 0$, transience also implies

$$\sup_{y\in V_{K'}} \mathrm{P}_{y,\omega}\left(|X_L| < K\right) \le \vartheta/9,$$

if L is large enough. Altogether, we have shown that for L sufficiently large,

$$\sup_{x:|x|\le L^3} \sup_{y\in V_L(x)} \mathrm{E}_{y,\omega}\left[\mathrm{E}_{X_L,\omega}\left[\overline{\tau}_{V_L(x)};\, T_l < \infty\right];\, \tau_{V_L(x)} > L\right] \le \vartheta.$$

Since ϑ can be chosen arbitrarily small, this shows (60) from which we deduce (59). □

10 Appendix

10.1 Some difference estimates

In this section we collect some difference estimates of (non)-smoothed exit distributions needed to prove Lemma 3.5 (i) and (iii). The first technical lemma compares the exit measure on ∂V_L of simple random walk to that on ∂C_L of standard Brownian motion.

Lemma 10.1. *Let $\beta, \eta > 0$ with $3\eta < \beta < 1$. For large L, there exists a constant $C = C(\beta,\eta) > 0$ such that for $A \subset \mathbb{R}^d$, $A^\beta = \{y \in \mathbb{R}^d : \mathrm{d}(y, A) \le L^\beta\}$ and $x \in V_L$ with $\mathrm{d}_L(x) > L^\beta$, the following holds.*

(i) $\pi_L(x, A) \le \pi_L^{\mathrm{BM}}\left(x, A^\beta\right)\left(1 + CL^{-(\beta-3\eta)}\right) + L^{-(d+1)}$.

(ii) $\pi_L^{\mathrm{BM}}(x, A) \le \pi_L(x, A^\beta)\left(1 + CL^{-(\beta-3\eta)}\right) + L^{-(d+1)}$.

Proof: (i) Set $L' = L + L^\eta$, $L'' = L + 2L^\eta$ and denote by $A^\beta_\prime$ the image of A^β on $\partial C_{L'}$ under the map $y \mapsto (L'/L)y$. With $\hat{x} = (L'/L)x$, using the Poisson kernel representation (8) in the second equality,

$$\pi_L^{\mathrm{BM}}\left(x, A^\beta\right) = \pi_{L'}^{\mathrm{BM}}\left(\hat{x}, A^\beta_\prime\right) = \int_{A^\beta_\prime} \frac{((L')^2 - |\hat{x}|^2)\,|x-y|^d}{((L')^2 - |x|^2)\,|\hat{x}-y|^d}\pi_{L'}^{\mathrm{BM}}(x, dy).$$

Since $|x| \le L + 1 - L^\beta$ and $0 < \eta < \beta < 1$, an evaluation of the integrand shows

$$\pi_L^{\mathrm{BM}}\left(x, A^\beta\right) \ge \pi_{L'}^{\mathrm{BM}}(x, A^\beta_\prime)\left(1 - C(\beta,\eta)L^{-(\beta-\eta)}\right) \tag{61}$$

for some positive constant $C(\beta,\eta)$. By [37], Corollary 1, for each $k \in \mathbb{N}$ there exists a constant $C_1 = C_1(k) > 0$ such that for each integer $n \ge 1$, one can construct on the same probability space a Brownian motion W_t with covariance matrix $d^{-1}I_d$ as well as simple random walk X_n, both starting in x and satisfying (with $\mathbb{Q}$ denoting the probability measure on that space)

$$\mathbb{Q}\left(\max_{0\le m\le n}|X_m - W_m| > C_1 \log n\right) \le C_1 n^{-k}. \tag{62}$$

Choose $k > (2/5)(d+1)$ and let $C_1(k)$ be the corresponding constant. The following arguments hold for sufficiently large L. By standard results on the oscillation of Brownian paths,

$$\mathbb{Q}\left(\sup_{0\le t\le L^{5/2}}\left|W_{\lfloor t\rfloor} - W_t\right| > (5/2)C_1\log L\right) \le (1/3)L^{-(d+1)}. \tag{63}$$

With

$$B_1 = \left\{\sup_{0\le t\le L^{5/2}}\left|X_{\lfloor t\rfloor} - W_t\right| \le 5C_1\log L\right\},$$

we deduce from (62) and (63) that

$$\mathbb{Q}\left(B_1^c\right) \leq (2/3)L^{-(d+1)}.$$

Let $\tau' = \inf\{t \geq 0 : W_t \notin C_{L'}\}$ and $B_2 = \{\tau' \vee \tau_{L''} \leq L^{5/2}\}$. We claim that

$$\mathbb{Q}\left(B_2^c\right) \leq (1/3)L^{-(d+1)}. \tag{64}$$

By the central limit theorem, one finds a constant $c > 0$ with $\mathbb{Q}\left(\tau_{L''} \leq (L'')^2\right) \geq c$ for L large. By the Markov property, we obtain $\mathbb{Q}\left(\tau_{L''} > L^{5/2}\right) \leq (1-c)^{L^{1/3}}$. A similar bound holds for the probability $\mathbb{Q}\left(\tau' > L^{5/2}\right)$, and (64) follows. Since π_L^{BM} is unchanged if the Brownian motion is replaced by a Brownian motion with covariance $d^{-1}I_d$, we have

$$\begin{aligned} \pi_{L'}^{\mathrm{BM}}(x, A_{\prime}^{\beta}) &\geq \mathbb{Q}\left(X_{\tau_L} \in A,\, W_{\tau'} \in A_{\prime}^{\beta}\right) \\ &\geq \mathrm{P}_x\left(X_{\tau_L} \in A\right) - \mathbb{Q}\left(X_{\tau_L} \in A,\, W_{\tau'} \notin A_{\prime}^{\beta},\, B_1 \cap B_2\right) - L^{-(d+1)}. \end{aligned} \tag{65}$$

Let $U = \left\{z \in \mathbb{Z}^d : \mathrm{d}(z, (\partial C_{L'} \backslash A_{\prime}^{\beta})) \leq 5C_1 \log L\right\}$. Then

$$\mathbb{Q}\left(X_{\tau_L} \in A,\, W_{\tau'} \notin A_{\prime}^{\beta},\, B_1 \cap B_2\right) \leq \mathrm{P}_x\left(X_{\tau_L} \in A,\, T_U < \tau_{L''}\right).$$

By the strong Markov property,

$$\mathrm{P}_x\left(X_{\tau_L} \in A,\, T_U < \tau_{L''}\right) \leq \mathrm{P}_x\left(X_{\tau_L} \in A\right) \sup_{y \in A} \mathrm{P}_y\left(T_U < \tau_{L''}\right).$$

Further, there exists a constant $c > 0$ such that for $y \in A$ and $z \in U$, we have $|y - z| \geq cL^{\beta}$ and $\mathrm{d}_{L''}(z) \leq \mathrm{d}_{L''}(y) \leq 2L^{\eta}$. Therefore, an application of first Lemma 3.2 (ii) and then Lemma 3.4 yields

$$\mathrm{P}_y\left(T_U < \tau_{L''}\right) \leq CL^{2\eta} \sum_{z \in U} \frac{1}{|y - z|^d} \leq CL^{2\eta}(\log L)L^{-\beta} \leq CL^{-(\beta - 3\eta)},$$

uniformly in $y \in A$. Going back to (65), we arrive at

$$\pi_{L'}^{\mathrm{BM}}\left(x, A_{\prime}^{\beta}\right) \geq \pi_L(x, A)\left(1 - CL^{-(\beta - 3\eta)}\right) - L^{-(d+1)}.$$

Together with (61), this shows (i).
(ii) The ideas are the same as in (i), so we only sketch the proof. Set $L' = L - L^{\eta}$, $L'' = L + L^{\eta}$. Denote by $A_{\prime}$ the image of A on $\partial C_{L'}$ under $y \mapsto (L'/L)y$. Similar to (61), one finds

$$\pi_L^{\mathrm{BM}}(x, A) \leq \pi_{L'}^{\mathrm{BM}}(x, A_{\prime})\left(1 + C(\beta, \eta)L^{-(\beta - \eta)}\right).$$

With B_1, B_2, τ' and W_t, $\mathbb{Q}$ defined as above, $\mathrm{P}_x^{\mathrm{BM}}$ the law of W_t conditioned on $W_0 = x$,

$$\pi_L(x, A^{\beta}) \geq \mathrm{P}_x^{\mathrm{BM}}\left(W_{\tau'} \in A_{\prime}\right) - \mathbb{Q}\left(W_{\tau'} \in A_{\prime},\, X_{\tau_L} \notin A^{\beta},\, B_1 \cap B_2\right) - L^{-(d+1)}.$$

Then, with $U = \left\{z \in \mathbb{R}^d : \mathrm{d}(z, (\partial C_L \backslash A^\beta)) \leq 5C_1 \log L\right\}$, $\tau'' = \inf\{t \geq 0 : W_t \notin C_{L''}\}$,

$$\mathbb{Q}\left(W_{\tau'} \in A_{\prime},\, X_{\tau_L} \notin A^\beta,\, B_1 \cap B_2\right) \leq \mathrm{P}_x^{\mathrm{BM}}\left(W_{\tau'} \in A_{\prime}\right) \sup_{y \in A_{\prime}} \mathrm{P}_y^{\mathrm{BM}}\left(T_U < \tau''\right).$$

Using the hitting estimates for Brownian motion from Lemma 3.3, one obtains for $y \in A_{\prime}$

$$\mathrm{P}_y^{\mathrm{BM}}\left(T_U < \tau''\right) \leq CL^{-(\beta - 3\eta)}.$$

Altogether, (ii) follows. □

We write $\hat{\pi}_\psi^{\mathrm{BM}}(x,z)$ for the density of $\hat{\pi}_\psi^{\mathrm{BM}}(x,dz)$ with respect to d-dimensional Lebesgue measure, i.e. for $\psi = (m_x)_{x \in \mathbb{R}^d}$,

$$\hat{\pi}_\psi^{\mathrm{BM}}(x,z) = \frac{1}{m_x}\varphi\left(\frac{|z-x|}{m_x}\right)\pi_{C_{|z-x|}}^{\mathrm{BM}}(0, z-x). \tag{66}$$

Lemma 10.2. *There exists a constant $C > 0$ such that for large L, $\psi = (m_y) \in \mathcal{M}_L$, $x, x' \in \{(2/3)L \leq |y| \leq (3/2)L\} \cap \mathbb{Z}^d$ and any $z, z' \in \mathbb{Z}^d$,*

(i) $\hat{\pi}_\psi(x,z) \leq CL^{-d}$.

(ii) $\hat{\pi}_\psi^{\mathrm{BM}}(x,z) \leq CL^{-d}$.

(iii) $|\hat{\pi}_\psi(x,z) - \hat{\pi}_\psi(x',z)| \leq C|x-x'|L^{-(d+1)}\log L$.

(iv) $|\hat{\pi}_\psi(x,z) - \hat{\pi}_\psi(x,z')| \leq C|z-z'|L^{-(d+1)}\log L$.

(v) $|\hat{\pi}_\psi^{\mathrm{BM}}(x,z) - \hat{\pi}_\psi^{\mathrm{BM}}(x',z)| \leq C|x-x'|L^{-(d+1)}$.

(vi) $|\hat{\pi}_\psi^{\mathrm{BM}}(x,z) - \hat{\pi}_\psi^{\mathrm{BM}}(x,z')| \leq C|z-z'|L^{-(d+1)}$.

(vii) $|\hat{\pi}_\psi(x,z) - \hat{\pi}_\psi^{\mathrm{BM}}(x,z)| \leq L^{-(d+1/4)}$.

Corollary 10.1. *In the situation of the preceding lemma,*

(i)

$$\begin{aligned} &|\hat{\pi}_\psi(x,z) - \hat{\pi}_\psi(x',z)| \\ &\quad\leq\; C\min\left\{|x-x'|L^{-(d+1)}\log L,\, |x-x'|L^{-(d+1)} + L^{-(d+1/4)}\right\}. \end{aligned}$$

(ii)

$$\begin{aligned} &|\hat{\pi}_\psi(x,z) - \hat{\pi}_\psi(x,z')| \\ &\quad\leq\; C\min\left\{|z-z'|L^{-(d+1)}\log L,\, |z-z'|L^{-(d+1)} + L^{-(d+1/4)}\right\}. \end{aligned}$$

Proof. Combine (iii)-(vii). □

Remark 10.1. The condition on x and x' in the lemma is only to ensure that both points lie in the domain of ψ.

Proof of Lemma 10.2: (i), (ii) This follows from the definition of $\hat{\pi}_\psi$, $\hat{\pi}_\psi^{\mathrm{BM}}$ together with Lemma 3.1 (i) and the explicit form of the Poisson kernel (8), respectively.
(iii), (iv) We can restrict ourselves to the case $|x-x'|=1$ as otherwise we take a shortest path connecting x with x' inside $\{(2/3)L \le |y| \le (3/2)L\}$ and apply the result for distance 1 $O(|x-x'|)$ times. We have

$$\begin{aligned}
&\hat{\pi}_\psi(x,z) - \hat{\pi}_\psi(x',z) \\
&= \left(1 - \frac{m_x}{m_{x'}}\right)\hat{\pi}_\psi(x,z) + \frac{1}{m_{x'}}\int_{\mathbb{R}_+}\left(\varphi\left(\frac{t}{m_x}\right) - \varphi\left(\frac{t}{m_{x'}}\right)\right)\pi_{V_t(x)}(x,z)\mathrm{d}t \\
&\quad + \frac{1}{m_{x'}}\int_{\mathbb{R}_+}\varphi\left(\frac{t}{m_{x'}}\right)\left(\pi_{V_t(x)}(x,z) - \pi_{V_t(x')}(x',z)\right)\mathrm{d}t \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

Using the fact that $\psi \in \mathcal{M}_L$ and part (i) for $\hat{\pi}_\psi(x,z)$, it follows that $|I_1| \le CL^{-(d+1)}$. Using additionally the smoothness of φ and, by Lemma 3.1 (i), $|\pi_{V_t(x)}(x,z)| \le CL^{-(d-1)}$, we also have $|I_2| \le CL^{-(d+1)}$. It remains to handle I_3. By translation invariance of simple random walk, $\pi_{V_t(x)}(x,z) = \pi_{V_t}(0,z-x)$. In particular, both (iii) and (iv) will follow if we prove that

$$\left|\int_{\mathbb{R}_+}\varphi\left(\frac{t}{m_x}\right)\left(\pi_{V_t}(0,z-x) - \pi_{V_t}(0,z-x')\right)\mathrm{d}t\right| \le CL^{-d}\log L \tag{67}$$

for x, x' with $|x-x'|=1$. By definition of $\mathcal{M}_L$, $m_x \in (L/10, 5L)$. We may therefore assume that $L/10 < |y-z| < 10L$ for $y = x, x'$. Due to the smoothness of φ and the fact that the integral is over an interval of length at most 2, (67) will follow if we show

$$\left|\int_{L/10}^{10L}\left(\pi_{V_t}(0,z-x) - \pi_{V_t}(0,z-x')\right)\mathrm{d}t\right| \le CL^{-d}\log L.$$

We set $J = \{t > 0 : z - x \in \partial V_t\}$ and $J' = \{t > 0 : z - x' \in \partial V_{t'}\}$, where

$$t' = t'(t) = \left|t\frac{(z-x)}{|z-x|} - (x'-x)\right|.$$

J is an interval of length at most 1, and J' has the same length up to order $O(L^{-1})$. Furthermore, $|J\Delta J'|$ is of order $O(L^{-1})$, and $\left|\frac{\mathrm{d}}{\mathrm{d}t}t'\right| = 1 + O(L^{-1})$. Using that both $\pi_{V_t}(0,z-x)$ and $\pi_{V_t}(0,z-x')$ are of order $O(L^{-(d-1)})$, it therefore suffices to prove

$$\left|\int_{J\cap J'}\left(\pi_{V_t(x)}(x,z) - \pi_{V_{t'}(x')}(x',z)\right)\mathrm{d}t\right| \le CL^{-d}\log L. \tag{68}$$

Write V for $V_t(x)$ and V' for $V_{t'}(x')$. By a first exit decomposition,

$$\pi_V(x,z) \le \pi_{V'}(x,z) + \sum_{y\in V\setminus V'} \mathrm{P}_x\left(T_y < \tau_V\right)\pi_V(y,z).$$

By Lemma 3.1 (ii), we can replace $\pi_{V'}(x,z)$ by $\pi_{V'}(x',z)+O(L^{-d})$. For $y \in V\backslash V'$ we have by Lemma 3.2 (ii) $\pi_V(y,z)=O(|y-z|^{-d})$ and $\mathrm{P}_x(T_y<\tau_V)=O(L^{-(d-1)})$, uniformly in $t \in J\cap J'$. Further, using $|x-x'|=1$, we have with $r=|z-x|$

$$\bigcup_{t\in J\cap J'}(V\backslash V')\subset V_r(x)\backslash V_{r-2}(x')\subset x+\mathrm{Sh}_r(3),$$

and for any $y\in \mathrm{Sh}_r(3)$, it follows by a geometric consideration that

$$\int_{J\cap J'} 1_{\{y\in V\backslash V'\}}\mathrm{d}t \le C\frac{|y-z|}{L}.$$

Altogether, applying Lemma 3.4 in the last step,

$$\begin{aligned}&\int_{J\cap J'}\pi_V(x,z)\mathrm{d}t\\ &\le \int_{J\cap J'}\pi_{V'}(x',z)\mathrm{d}t+O(L^{-d})+CL^{-(d-1)}\sum_{y\in x+\mathrm{Sh}_r(3)}\frac{1}{|y-z|^d}\frac{|y-z|}{L}\\ &\le \int_{J\cap J'}\pi_{V'}(x',z)\mathrm{d}t+CL^{-d}\log L.\end{aligned}$$

The reverse inequality, proved in the same way, then implies (68).
(v) We can assume $|x-x'|\le 1$. Then the claim follows from

$$\begin{aligned}&\left|\hat{\pi}_\psi^{\mathrm{BM}}(x,z)-\hat{\pi}_\psi^{\mathrm{BM}}(x',z)\right|\\ &=\frac{1}{d\,alpha}\left|\frac{1}{m_x}\varphi\left(\frac{|z-x|}{m_x}\right)\frac{1}{|z-x|^{d-1}}-\frac{1}{m_{x'}}\varphi\left(\frac{|z-x'|}{m_{x'}}\right)\frac{1}{|z-x'|^{d-1}}\right|.\end{aligned}$$

(vi) This is proved in the same way as (v).
(vii) Fix $\alpha=2/3$, $\beta=1/3$, and let $0<\eta<1/40$. Set $A=C_{L^\alpha}(z)$ and $A^{\mathbb{Z}}=A\cap\mathbb{Z}^d$. By part (iv), we have

$$\hat{\pi}_\psi(x,z)\le\frac{1}{|A^{\mathbb{Z}}|}\hat{\pi}_\psi\left(x,A^{\mathbb{Z}}\right)+CL^{-(d+1-\alpha)}\log L. \tag{69}$$

Further,

$$\hat{\pi}_\psi\left(x,A^{\mathbb{Z}}\right)=\frac{1}{m_x}\int_{L/10}^{10L}\varphi\left(\frac{t}{m_x}\right)\pi_{V_t(x)}\left(x,A^{\mathbb{Z}}\right)\mathrm{d}t. \tag{70}$$

By Lemma 10.1 (i), it follows that for $t\in(L/10,10L)$

$$\pi_{V_t(x)}\left(x,A^{\mathbb{Z}}\right)\le\pi_{V_t(x)}^{\mathrm{BM}}\left(x,A^\beta\right)\left(1+CL^{-(\beta-3\eta)}\right)+CL^{-(d+1)},$$

where $A^\beta=C_{L^\alpha+L^\beta}(z)$ and the constant C is uniform in t. If we plug the last line into (70) and use part (ii) and (vi), we arrive at

$$\begin{aligned}\hat{\pi}_\psi\left(x,A^{\mathbb{Z}}\right)&\le\hat{\pi}_\psi^{\mathrm{BM}}(x,A^\beta)\left(1+CL^{-(\beta-3\eta)}\right)+CL^{-(d+1)}\\ &\le\hat{\pi}_\psi^{\mathrm{BM}}(x,A)\left(1+CL^{-(\beta-3\eta)}\right)+CL^{-d}L^{(d-1)\alpha+\beta}\\ &\le|A|\cdot\hat{\pi}_\psi^{\mathrm{BM}}(x,z)+CL^{d\alpha}L^{-(d+\beta-3\eta)}.\end{aligned}$$

Notice that in our notation, $|A|$ is the volume of A, while $|A^{\mathbb{Z}}|$ is the cardinality of $A^{\mathbb{Z}}$. From *Gauss* we have learned that $|A| = |A^{\mathbb{Z}}| + O\left(L^{(d-1)\alpha}\right)$. Going back to (69), this implies

$$\hat{\pi}_\psi(x,z) \le \hat{\pi}_\psi^{\mathrm{BM}}(x,z) + L^{-(d+1/4)},$$

as claimed. To prove the reverse inequality, we can follow the same steps, replacing the random walk estimates by those of Brownian motion and vice versa. □

10.2 Proof of Lemma 3.5

Proof of Lemma 3.5: (i) Set $\alpha = 2/3$, $\beta = 1/3$ and $\eta = \mathrm{d}(x, \partial V_L)$. Choose $y_1 \in \partial V_L$ such that $|x - y_1| = \eta$. First assume $\eta \le L^\beta$. The following estimates are valid for L large. Write

$$\phi_{L,\psi}(x,z) = \sum_{\substack{y\in\partial V_L:\\ |y-y_1|\le L^\alpha}} \pi_L(x,y)\hat{\pi}_\psi(y,z) + \sum_{\substack{y\in\partial V_L:\\ |y-y_1|> L^\alpha}} \pi_L(x,y)\hat{\pi}_\psi(y,z) = I_1 + I_2.$$

For I_2, notice that $|y - y_1| > L^\alpha$ implies $|y - x| > L^\alpha/2$. Using Lemmata 10.2 (i), 3.2 (iii) in the first and Lemma 3.4 in the second inequality, we have

$$I_2 \le C\eta L^{-d} \sum_{\substack{y\in\partial V_L:\\ |y-y_1|> L^\alpha}} \frac{1}{|x-y|^d} \le C\eta L^{-(d+\alpha)} \le L^{-(d+1/4)}. \tag{71}$$

For I_1, we first use Lemma 10.2 part (iii) to deduce

$$\hat{\pi}_\psi(y,z) \le \hat{\pi}_\psi(y_1,z) + CL^{-(d+1-\alpha)}\log L.$$

Therefore by part (vii),

$$I_1 \le \hat{\pi}_\psi(y_1,z) + L^{-(d+1/4)} \le \hat{\pi}_\psi^{\mathrm{BM}}(y_1,z) + 2L^{-(d+1/4)}.$$

From the Poisson formula (8) we deduce much as in (71) that

$$\int\limits_{y\in\partial C_L:|y-y_1|>L^\alpha} \pi_L^{\mathrm{BM}}(x,dy) \le L^{-1/4}.$$

Using Lemma 10.2 (ii) in the first and (v) in the second inequality, we conclude that

$$\begin{aligned}
\hat{\pi}_\psi^{\mathrm{BM}}(y_1,z) &\le \hat{\pi}_\psi^{\mathrm{BM}}(y_1,z) \int\limits_{y\in\partial C_L:|y-y_1|\le L^\alpha} \pi_L^{\mathrm{BM}}(x,dy) + CL^{-(d+1/4)} \\
&\le \int\limits_{y\in\partial C_L:|y-y_1|\le L^\alpha} \pi_L^{\mathrm{BM}}(x,dy)\hat{\pi}_\psi^{\mathrm{BM}}(y,z) + CL^{-(d+1/4)} \\
&\le \phi_{L,\psi}^{\mathrm{BM}}(x,z) + CL^{-(d+1/4)}.
\end{aligned}$$

Now we look at the case $\eta > L^\beta$. We take a cube U_1 of radius L^α, centered at y_1, and set $W_1 = \partial V_L \cap U_1$. Then we can find a partition of $\partial V_L \backslash W_1$ into disjoint sets $W_i = \partial V_L \cap U_i$, $i = 2, \ldots, k_L$, where U_i is a cube such that for some $c_1, c_2 > 0$ depending only on d,

$$c_1 L^{\alpha(d-1)} \leq |W_i| \leq c_2 L^{\alpha(d-1)}.$$

For $i \geq 2$, we fix an arbitrary $y_i \in W_i$. Let $W_i^\beta = \{y \in \mathbb{R}^d : \mathrm{d}(y, W_i) \leq L^\beta\}$. Applying first Lemma 10.2 (iii) and then Lemma 10.1 (i) gives

$$\begin{aligned} \phi_{L,\psi}(x,z) &\leq \sum_{i=1}^{k_L} \pi_L(x, W_i) \hat{\pi}_\psi(y_i, z) + L^{-(d+1/4)} \\ &\leq \sum_{i=1}^{k_L} \pi_L^{\mathrm{BM}}(x, W_i^\beta) \hat{\pi}_\psi(y_i, z) \left(1 + L^{-1/4}\right) + L^{-(d+1/4)}. \end{aligned} \tag{72}$$

As the W_i^β overlap, we refine them as follows: Set $\tilde{W}_1 = W_1^\beta \cap \partial C_L$, and split $\partial C_L \backslash \tilde{W}_1$ into a collection of disjoint measurable sets $\tilde{W}_i \subset \partial C_L \cap W_i^\beta$, $i = 2, \ldots, k_L$, such that $\cup_{i=1}^{k_L} \tilde{W}_i = \partial C_L$ and $|(W_i^\beta \cap \partial C_L) \backslash \tilde{W}_i| \leq C_1 L^{\alpha(d-2)+\beta}$ for some $C_1 = C_1(d)$. By construction we can find constants $c_3, c_4 > 0$ such that $|\tilde{W}_i| \geq c_3 L^{\alpha(d-1)}$ and, for $i = 2, \ldots, k_L$,

$$\inf_{y \in W_i^\beta} |x - y| \geq c_4 \sup_{y \in \tilde{W}_i} |x - y|,$$

which implies by (8) that

$$\sup_{y \in W_i^\beta} \pi_L^{\mathrm{BM}}(x, y) \leq c_4^{-1} \inf_{y \in \tilde{W}_i} \pi_L^{\mathrm{BM}}(x, y).$$

For $i = 1, \ldots, k_L$ we then have

$$\pi_L^{\mathrm{BM}}(x, W_i^\beta) \leq \pi_L^{\mathrm{BM}}(x, \tilde{W}_i) \left(1 + C_1 c_3^{-1} L^{\beta - \alpha}\right) \leq \pi_L^{\mathrm{BM}}(x, \tilde{W}_i) \left(1 + L^{-1/4}\right).$$

Plugging the last line into (72),

$$\phi_{L,\psi}(x,z) \leq \sum_{i=1}^{k_L} \pi_L^{\mathrm{BM}}(x, \tilde{W}_i) \hat{\pi}_\psi(y_i, z) \left(1 + L^{-1/4}\right) + L^{-(d+1/4)}.$$

A reapplication of Lemma 10.2 (iii), (vii) and then (ii) yields

$$\begin{aligned} \phi_{L,\psi}(x,z) &\leq \sum_{i=1}^{k_L} \int_{\tilde{W}_i} \pi_L^{\mathrm{BM}}(x, dy) \hat{\pi}_\psi(y, z) + L^{-(d+1/4)} \\ &\leq \sum_{i=1}^{k_L} \int_{\tilde{W}_i} \pi_L^{\mathrm{BM}}(x, dy) \hat{\pi}_\psi^{\mathrm{BM}}(y, z) \left(1 + L^{-1/4}\right) + L^{-(d+1/4)} \\ &= \phi_{L,\psi}^{\mathrm{BM}}(x,z) + C L^{-(d+1/4)}. \end{aligned}$$

The reverse inequality in both the cases $\eta \leq L^\beta$ and $\eta > L^\beta$ is obtained similarly.
(ii) Let $\psi = (m_y)_y \in \mathcal{M}_L$ and $z \in \mathbb{Z}^d$. For $y \in \mathbb{R}^d$ with $L/2 < |y| < 2L$ we set

$$g(y,z) = \frac{1}{m_y}\varphi\left(\frac{|z-y|}{m_y}\right)\pi^{\mathrm{BM}}_{C_{|z-y|}}(0, z-y). \tag{73}$$

Then

$$\phi^{\mathrm{BM}}_{L,\psi}(x,z) = \int_{\partial C_L} \pi^{\mathrm{BM}}_L(x,dy) g(y,z).$$

Choose a cutoff function $\chi \in C^\infty\left(\mathbb{R}^d\right)$ with compact support in $\{x \in \mathbb{R}^d : 1/2 < |x| < 2\}$ such that $\chi \equiv 1$ on $\{2/3 \leq |x| \leq 3/2\}$. Setting $m_v = 1$ for $v \notin \{L/2 < |x| < 2L\}$, we define

$$\tilde{g}(y,z) = g(Ly,z)\chi(y), \quad y \in \mathbb{R}^d.$$

By (8) we have the representation

$$\tilde{g}(y,z) = \frac{1}{d\,\alpha\, m_{Ly}}|z-Ly|^{-d+1}\varphi\left(\frac{|z-Ly|}{m_{Ly}}\right)\chi(y).$$

Notice that $\tilde{g}(\cdot,z) \in C^4\left(\mathbb{R}^d\right)$, with $\tilde{g}(y,z) = 0$ if $|z-Ly| \notin (L/5, 10L)$ or $|y| \notin (1/2, 2)$. The Poisson integral $u(\overline{x},z) = \phi^{\mathrm{BM}}_{L,\psi}(x,z),\ x = L\overline{x}$, solves the Dirichlet problem

$$\begin{cases} \Delta_{\overline{x}} u(\overline{x},z) = 0 \quad\quad , & \overline{x} \in C_1 \\ \quad u(\overline{x},z) = \tilde{g}(\overline{x},z), & \overline{x} \in \partial C_1 \end{cases}. \tag{74}$$

where $\Delta_{\overline{x}}$ is the Laplace operator with respect to $\overline{x}$. Moreover, by Corollary 6.5.4 of Krylov [21], $u(\cdot,z)$ is smooth on $\overline{C}_1$. Write

$$|u(\cdot,z)|_k = \sum_{i=0}^k \left|\left|D^i u(\cdot,z)\right|\right|_{C_1}.$$

Theorem 6.3.2 in the same book shows that for some $C > 0$ independent of z

$$|u(\cdot,z)|_3 \leq C|\tilde{g}(\cdot,z)|_4.$$

A direct calculation shows that $\sup_{z\in\mathbb{R}^d} |\tilde{g}(\cdot,z)|_4 \leq CL^{-d}$. Now the claim follows from

$$\left|\left|D^i\phi^{\mathrm{BM}}_{L,\psi}(\cdot,z)\right|\right|_{C_L} = L^{-i}\left|\left|D^i u(\cdot,z)\right|\right|_{C_1}.$$

(iii) Let $x, x' \in V_L \cup \partial V_L$. Choose $\tilde{x} \in V_L$ next to x and $\tilde{x}' \in V_L$ next to x'. Then $|\tilde{x} - x| = 1$ if $x \in \partial V_L$ and $\tilde{x} = x$ otherwise. By the triangle inequality,

$$\begin{aligned} &|\phi_{L,\psi}(x,z) - \phi_{L,\psi}(x',z)| \\ &\quad \leq \ |\phi_{L,\psi}(x,z) - \phi_{L,\psi}(\tilde{x},z)| + |\phi_{L,\psi}(\tilde{x},z) - \phi_{L,\psi}(\tilde{x}',z)| + |\phi_{L,\psi}(\tilde{x}',z) - \phi_{L,\psi}(x',z)|\,. \end{aligned} \tag{75}$$

By parts (i) and (ii) combined with the mean value theorem, we get for the middle term

$$\begin{aligned}
&|\phi_{L,\psi}(\tilde{x},z) - \phi_{L,\psi}(\tilde{x}',z)| \\
&\leq \left|\phi_{L,\psi}(\tilde{x},z) - \phi^{\mathrm{BM}}_{L,\psi}(\tilde{x},z)\right| + \left|\phi^{\mathrm{BM}}_{L,\psi}(\tilde{x},z) - \phi^{\mathrm{BM}}_{L,\psi}(\tilde{x}',z)\right| + \left|\phi^{\mathrm{BM}}_{L,\psi}(\tilde{x}',z) - \phi_{L,\psi}(\tilde{x}',z)\right| \\
&\leq C\left(L^{-(d+1/4)} + |x-x'|L^{-(d+1)}\right).
\end{aligned}$$

If $x \in \partial V_L$, then $\phi_{L,\psi}(x,z) = \hat{\pi}_\psi(x,z)$, so that we can write the first term of (75) as

$$|\phi_{L,\psi}(x,z) - \phi_{L,\psi}(\tilde{x},z)| = \left|\sum_{y\in\partial V_L} \pi_L(\tilde{x},y)\left(\hat{\pi}_\psi(y,z) - \hat{\pi}_\psi(x,z)\right)\right|.$$

Set $A = \{y \in \partial V_L : |x-y| > L^{1/4}\}$. Then by Lemmata 3.2 (iii) and 3.4,

$$\pi_L(\tilde{x},A) \leq C\sum_{y\in A}\frac{1}{|x-y|^d} \leq CL^{-1/4}.$$

For all $y \in \partial V_L$, we have by Lemma 10.2 (i) that $|\hat{\pi}_\psi(y,z) - \hat{\pi}_\psi(x,z)| \leq CL^{-d}$. If $y \in \partial V_L \backslash A$, then part (iii) gives $|\hat{\pi}_\psi(y,z) - \hat{\pi}_\psi(x,z)| \leq CL^{-(d+3/4)}\log L$. Altogether,

$$|\phi_{L,\psi}(x,z) - \phi_{L,\psi}(\tilde{x},z)| \leq CL^{-(d+1/4)}.$$

The third term of (75) is treated in exactly the same way. $\square$

10.3 Proof of Lemma 3.2

We start with an auxiliary lemma, which already includes the upper bound of part (iii).

Lemma 10.3. *Let $x \in V_L$, $y \in \partial V_L$, and set $t = |x-y|$.*

(i)
$$\mathrm{P}_x\left(X_{\tau_L} = y\right) \leq C\,\mathrm{d}_L(x)^{-d+1}.$$

(ii)
$$\mathrm{P}_x(X_{\tau_L} = y) \leq C\frac{\max\{1,\mathrm{d}_L(x)\}}{|x-y|}\max_{x'\in\partial V_{t/3}(y)\cap V_L}\mathrm{P}_{x'}(X_{\tau_L} = y).$$

(iii)
$$\mathrm{P}_x(X_{\tau_L} = y) \leq C\frac{\max\{1,\mathrm{d}_L(x)\}}{|x-y|^d}.$$

Proof: (i) We can assume that $s = \mathrm{d}_L(x) \geq 6$. If $s' = \lfloor s/3 \rfloor$, then $\partial V_{s'}(x) \subset V_{L-s'}$. Using Lemma 3.1 (iii), we compute for any $y' \in V_L$ with $|y-y'| = 1$,

$$\mathrm{P}_{y'}\left(T_{\partial V_{s'}(x)} < \tau_L\right) \leq \mathrm{P}_{y'}\left(T_{V_{L-s'}} < \tau_L\right) \leq Cs^{-1}.$$

By Lemma 3.2 (i) it follows that uniformly in $z \in \partial V_{s'}(x)$,

$$\mathrm{P}_z\left(T_x < \tau_L\right) \le \mathrm{P}_z\left(T_x < \infty\right) \le C(s')^{-d+2} \le Cs^{-d+2}.$$

Thus, by the strong Markov property at $T_{\partial V_{s'}(x)}$,

$$\mathrm{P}_{y'}\left(T_x < \tau_L\right) \le Cs^{-d+1}.$$

Since by time reversibility of simple random walk

$$\mathrm{P}_x\left(X_{\tau_L} = y\right) = \sum_{\substack{y' \in V_L, \\ |y'-y|=1}} \mathrm{P}_x\left(X_{\tau_L} = y,\, X_{\tau_L - 1} = y'\right) = \frac{1}{2d} \sum_{\substack{y' \in V_L, \\ |y'-y|=1}} \mathrm{P}_{y'}\left(T_x < \tau_L\right),$$

the claim is proved.
(ii) We may assume that $t = |x-y| > 100d$ and $\mathrm{d}_L(x) < t/100$. Choose a point x' outside V_L such that $V_{t/10}(x') \cap V_L = \emptyset$ and $|x - x'| \le \mathrm{d}_L(x) + t/10 + \sqrt{d}$. Then $|x - x'| \le t/5$. Furthermore, since $|x' - y| \ge 4t/5$,

$$\left(V_{t/4}(x') \cup \partial V_{t/4}(x')\right) \cap V_{t/3}(y) = \emptyset.$$

We apply twice the strong Markov property and obtain

$$\mathrm{P}_x(X_{\tau_L} = y) \le \mathrm{P}_x\left(\tau_{V_{t/4}(x')} < T_{V_{t/10}(x')}\right) \max_{z \in \partial V_{t/3}(y) \cap V_L} \mathrm{P}_z(X_{\tau_L} = y).$$

Evaluating the expression in Lemma 3.1 (iii) shows

$$\mathrm{P}_x\left(\tau_{V_{t/4}(x')} < T_{V_{t/10}(x')}\right) \le C\frac{\max\{1, \mathrm{d}_L(x)\}}{t},$$

which concludes the proof of part (ii).
(iii) By (ii) it suffices to prove that for some constant K and for all $l \ge 1$

$$\max_{z \in \partial V_{l/3}(y) \cap V_L} \mathrm{P}_z(X_{\tau_L} = y) \le Kl^{-d+1}. \tag{76}$$

Let c_1 and c_2 be the constants from (i) and (ii), respectively. Define $\eta = 3^{-d}c_2^{-1}$ and $K = \max\{3^{d(d-1)}c_2^{d-1}, c_1\eta^{-d+1}\}$. For $l \le 3^d c_2$ there is nothing to prove since $Kl^{-d+1} \ge 1$. Thus let $l > 3^d c_2$, and choose l_0 with $l_0 < l \le 2l_0$. Assume that (76) is proved for all $l' \le l_0$. We show that (76) also holds for l. For z with $\mathrm{d}_L(z) \ge \eta l$, it follows from (i) that

$$\mathrm{P}_z(X_{\tau_L} = y) \le c_1\eta^{-d+1}l^{-d+1} \le Kl^{-d+1}.$$

If $1 \le \mathrm{d}_L(z) < \eta l$, then by (ii) and the fact that $l/3 \le l_0$

$$\begin{aligned} \mathrm{P}_z(X_{\tau_L} = y) &\le c_2\frac{\max\{1, \mathrm{d}_L(z)\}}{|z-y|} \max_{z' \in \partial V_{l/9}(y) \cap V_L} \mathrm{P}_z(X_{\tau_L} = y) \\ &\le c_2 3\eta K\,(l/3)^{-d+1} \le Kl^{-d+1}. \end{aligned}$$

If $\mathrm{d}_L(z) < 1$, then again by (i)

$$\mathrm{P}_z(X_{\tau_L} = y) \leq c_2 3 l^{-1} K \,(l/3)^{-d+1} \leq K l^{-d+1}.$$

This proves the claim. □

Proof of Lemma 3.2: (i) follows from Proposition 6.4.2 of [24].
(ii) We consider different cases. If $|x-y| \leq \mathrm{d}_L(y)/2$, then $\mathrm{d}_L(x) \geq \mathrm{d}_L(y)/2$ and thus by Lemma 3.2 (i)

$$\mathrm{P}_x\left(T_{V_a(y)} < \tau_L\right) \leq \mathrm{P}_x\left(T_{V_a(y)} < \infty\right) \leq C\left(\frac{a}{|x-y|}\right)^{d-2} \leq C\frac{a^{d-2}\,\mathrm{d}_L(y)\,\mathrm{d}_L(x)}{|x-y|^d}.$$

For the rest of the proof we assume that $|x-y| > \mathrm{d}_L(y)/2$. Set $a' = \mathrm{d}_L(y)/5$. First we argue that in the case $1 \leq a \leq a'$, we only have to prove the bound for a'. Indeed, if $\mathrm{d}_L(y)/6 \leq a < a'$, we get an upper bound by replacing a by a'. For $1 \leq a < \mathrm{d}_L(y)/6$, the strong Markov property together with Lemma 3.2 (i) yields

$$\begin{aligned}
\mathrm{P}_x\left(T_{V_a(y)} < \tau_L\right) &\leq \max_{z\in\partial(\mathbb{Z}^d\setminus V_{a'}(y))} \mathrm{P}_z\left(T_{V_a(y)} < \tau_L\right)\mathrm{P}_x\left(T_{V_{a'}(y)} < \tau_L\right)\\
&\leq C\left(\frac{a}{a'-1}\right)^{d-2}\frac{(a')^{d-2}\,\mathrm{d}_L(y)\max\{1,\mathrm{d}_L(x)\}}{|x-y|^d}\\
&\leq C\frac{a^{d-2}\,\mathrm{d}_L(y)\max\{1,\mathrm{d}_L(x)\}}{|x-y|^d}.
\end{aligned}$$

Now we prove the claim for $a = \mathrm{d}_L(y)/5$. We take a point $y' \in \partial V_L$ closest to y. If $|x-z| \geq |x-y|/2$ for all $z \in V_a(y')$, then by Lemma 10.3 (iii)

$$\max_{z\in V_a(y')} \mathrm{P}_x\left(X_{\tau_L} = z\right) \leq C\,2^d\frac{\max\{1,\mathrm{d}_L(x)\}}{|x-y|^d}.$$

As a subset of $\mathbb{Z}^d$, $V_a(y') \cap \partial V_L$ contains on the order of $\mathrm{d}_L(y)^{d-1}$ points. Therefore, by Lemma 3.1 (i), we deduce that there exists some $\delta > 0$ such that

$$\min_{x'\in V_a(y)} \mathrm{P}_{x'}\left(X_{\tau_L} \in V_a(y')\right) \geq \delta.$$

We conclude that

$$\begin{aligned}
\frac{a^{d-1}\max\{1,\mathrm{d}_L(x)\}}{|x-y|^d} &\geq c\,\mathrm{P}_x\left(X_{\tau_L} \in V_a(y')\right) \geq c\,\mathrm{P}_x\left(X_{\tau_L} \in V_a(y'),\, T_{V_a(y)} < \tau_L\right)\\
&= c\sum_{x'\in V_a(y)} \mathrm{P}_x\left(X_{T_{V_a(y)}} = x',\, T_{V_a(y)} < \tau_L\right)\mathrm{P}_{x'}\left(X_{\tau_L} \in V_a(y')\right)\\
&\geq c\,\delta\cdot\mathrm{P}_x\left(T_{V_a(y)} < \tau_L\right). \qquad (77)
\end{aligned}$$

On the other hand, if $|x-z| < |x-y|/2$ for some $z \in V_a(y')$, then

$$|x-y| \leq |x-z| + |z-y'| + |y'-y| \leq 2\,\mathrm{d}_L(y) + |x-y|/2$$

and thus

$$\mathrm{d}_L(y)/2 < |x-y| \le 4\,\mathrm{d}_L(y). \tag{78}$$

If $\mathrm{d}_L(x) \ge 4\,\mathrm{d}_L(y)/5$, we use Lemma 3.2 (i) again. For $\mathrm{d}_L(x) < 4\,\mathrm{d}_L(y)/5$, we get by Lemma 3.1 (iii)

$$\mathrm{P}_x\left(T_{V_a(y)} < \tau_L\right) \le \mathrm{P}_x\left(T_{V_{L-4\,\mathrm{d}_L(y)/5}} < \tau_L\right) \le C\frac{\max\{1, \mathrm{d}_L(x)\}}{\mathrm{d}_L(y)}.$$

Together with (78), this proves the claim in this case. Altogether, we have proved the bound for $1 \le a \le \mathrm{d}_L(y)/5$. It remains to handle the case $\max\{1, \mathrm{d}_L(y)/5\} \le a$. If $z \in V_{6a}(y)$, we have that

$$|x-y| \le |x-z| + 6a$$

and thus, using $|x-y| > 7a$,

$$|x-y| \le 7|x-z|.$$

Therefore Lemma 10.3 (iii) yields

$$\max_{z\in V_{6a}(y)} \mathrm{P}_x\left(X_{\tau_L} = z\right) \le C\frac{\max\{1, \mathrm{d}_L(x)\}}{|x-z|^d} \le 7^d C\frac{\max\{1, \mathrm{d}_L(x)\}}{|x-y|^d}.$$

Again by Lemma 3.1 (i), we find some $\delta > 0$ such that

$$\min_{x'\in V_a(y)} \mathrm{P}_{x'}\left(X_{\tau_L} \in V_{6a}(y)\right) \ge \delta.$$

A similar argument to (77), with $V_a(y')$ replaced by $V_{6a}(y)$, finishes the proof of (ii).
(iii) It only remains to prove the lower bound. Let $t = |x-y|$. First assume $t \ge L/2$. Then Lemma 3.1 (iii) gives

$$\mathrm{P}_x\left(T_{V_{2L/3}} < \tau_L\right) \ge c\frac{\mathrm{d}_L(x)}{t},$$

and the claim follows from the strong Markov property and Lemma 3.1 (i). Now assume $t < L/2$. Let $x' \in V_L$ such that $V_t(x') \subset V_L$ and $y \in \partial V_t(x')$. If $\mathrm{d}_L(x) > t/2$, there is by Lemma 3.1 (i) a strictly positive probability to exit the ball $V_{t/2}(x)$ within $V_{2t/3}(x')$. Since by the same lemma,

$$\inf_{z\in V_{2t/3}(x')} \mathrm{P}_z\left(\tau_L = y\right) \ge ct^{-(d-1)}, \tag{79}$$

we obtain the claim in this case again by applying the strong Markov property. Finally, assume $\mathrm{d}_L(x) \le t/2$. Then a careful evaluation of the expression in Lemma 3.1 (iii) shows

$$\mathrm{P}_x\left(T_{V_{L-2t/3}} < \tau_L\right) \ge c\frac{\mathrm{d}_L(x)}{t},$$

and

$$\begin{aligned}
\mathrm{P}_x(\tau_L = y) &\geq \mathrm{P}_x\Big(\tau_L = y,\, T_{L-2t/3} < \tau_L,\, T_{V_{2t/3}(x')} < \tau_L\Big) \\
&\geq c\frac{\mathrm{d}_L(x)}{t}\,\mathrm{P}_x\Big(\tau_L = y \,|\, T_{L-2t/3} < \tau_L,\, T_{V_{2t/3}(x')} < \tau_L\Big) \\
&\quad \times \mathrm{P}_x\Big(T_{V_{2t/3}(x')} < \tau_L \,|\, T_{L-2t/3} < \tau_L\Big).
\end{aligned}$$

By a simple geometric consideration and again Lemma 3.1 (i), the second probability on the right side is bounded from below by some $\delta > 0$, and the first probability has already been estimated in (79). □

10.4 Proofs of Propositions 4.1 and 4.2

Since $\hat{\pi}_m(x, y) = \hat{\pi}_m(0, y - x)$, it suffices to look at $\hat{\pi}_m(x) = \hat{\pi}_m(0, x)$ and $\hat{g}_{m,\mathbb{Z}^d}(x) = \hat{g}_{m,\mathbb{Z}^d}(0, x)$. Recall the definition of γ_m from Section 4.1.

Proof of Proposition 4.1: For bounded m, that is $m \leq m_0$ for some m_0, the result is a special case of [24], Theorem 2.1.1. Also, for $n \leq n_0$ and all m, the statement follows from Lemma 10.2 (i). We therefore have to prove the proposition only for large n and m. To this end, let $B_m = [-\sqrt{\gamma_m}\,\pi,\, \sqrt{\gamma_m}\,\pi]^d$, and for $\theta \in B_m$ set

$$\phi_m(\theta) = \sum_{y \in \mathbb{Z}^d} e^{i\theta \cdot y/\sqrt{\gamma_m}}\hat{\pi}_m(y).$$

The Fourier inversion formula gives

$$\hat{\pi}_m^n(x) = \frac{1}{(2\pi)^d \gamma_m^{d/2}} \int_{B_m} e^{-ix\cdot\theta/\sqrt{\gamma_m}}[\phi_m(\theta)]^n \mathrm{d}\theta.$$

We decompose the integral into

$$(2\pi)^d \gamma_m^{d/2} n^{d/2} \hat{\pi}_m^n(x) = I_0(n, m, x) + \ldots + I_3(n, m, x),$$

where, with $\beta = \sqrt{n}\,\theta$,

$$\begin{aligned}
I_0(n, m, x) &= \int_{\mathbb{R}^d} e^{-ix\cdot\beta/\sqrt{n\gamma_m}} e^{-|\beta|^2/2}\mathrm{d}\beta, \\
I_1(n, m, x) &= \int_{|\beta| \leq n^{1/4}} e^{-ix\cdot\beta/\sqrt{n\gamma_m}} \Big([\phi_m(\beta/\sqrt{n})]^n - e^{-|\beta|^2/2}\Big)\mathrm{d}\beta, \\
I_2(n, m, x) &= -\int_{|\beta| > n^{1/4}} e^{-ix\cdot\beta/\sqrt{n\gamma_m}} e^{-|\beta|^2/2}\mathrm{d}\beta, \\
I_3(n, m, x) &= n^{d/2} \int_{n^{-1/4} < |\theta|,\, \theta \in B_m} e^{-ix\cdot\theta/\sqrt{\gamma_m}}[\phi_m(\theta)]^n \mathrm{d}\theta.
\end{aligned}$$

By completing the square in the exponential, we get

$$I_0(n,m,x) = (2\pi)^{d/2} \exp\left(-\frac{|x|^2}{2n\gamma_m}\right).$$

For I_1 and $|\beta| \le n^{1/4}$, we expand ϕ_m in a series around the origin,

$$\begin{aligned} \phi_m(\beta/\sqrt{n}) &= 1 - |\beta|^2/2n + |\beta|^4 O\left(n^{-2}\right), \\ \log\phi_m(\beta/\sqrt{n}) &= -|\beta|^2/2n + |\beta|^4 O\left(n^{-2}\right). \end{aligned} \tag{80}$$

Therefore,

$$[\phi_m(\beta/\sqrt{n})]^n = e^{-|\beta|^2/2}\left(1 + |\beta|^4 O\left(n^{-1}\right)\right),$$

so that

$$|I_1(n,m,x)| \le O\left(n^{-1}\right) \int\limits_{|\beta|\le n^{1/4}} e^{-|\beta|^2/2}|\beta|^4 \mathrm{d}\beta = O\left(n^{-1}\right).$$

Similarly, I_2 is bounded by

$$|I_2(n,m,x)| \le C\int_{n^{1/4}}^{\infty} r^{d-1}e^{-r^2/2}\mathrm{d}r = O\left(n^{-1}\right).$$

Concerning I_3, we follow closely [6], proof of Proposition B1, and split the integral further into

$$\begin{aligned} n^{-d/2}I_3(n,m,x) &= \int\limits_{n^{-1/4}<|\theta|\le a} + \int\limits_{a<|\theta|\le A} + \int\limits_{A<|\theta|\le m^\alpha} + \int\limits_{m^\alpha<|\theta|,\,\theta\in B_m} \\ &= \left(I_{3,0} + I_{3,1} + I_{3,2} + I_{3,3}\right)(n,m,x), \end{aligned}$$

where $0 < a < A$ and $\alpha \in (0,1)$ are constants that will be chosen in a moment, independently of n and m. By (80), we can find $a > 0$ such that for $|\beta| \le a\sqrt{n}$, $\log\phi_m(\theta) \le -|\theta|^2/3$ (recall that $\beta = \sqrt{n}\,\theta$). Then

$$|I_{3,0}(n,m,x)| \le C\int_{n^{-1/4}}^{\infty} r^{d-1}e^{-nr^2/3}\mathrm{d}r = O\left(n^{-(d+2)/2}\right).$$

As a consequence of Lemma 3.1 (i) and of our coarse graining, it follows that for any $0 < a < A$, one has for some $0 < \rho = \rho(a,A) < 1$, uniformly in m,

$$\sup_{a\le|\theta|\le A} |\phi_m(\theta)| \le \rho.$$

Using this fact,

$$|I_{3,1}(n,m,x)| \le CA^d\rho^n = O\left(n^{-(d+2)/2}\right).$$

To deal with the last two integrals is more delicate since we have to take into account the m-dependency. First,

$$|I_{3,2}(n,m,x)| \le \int\limits_{A<|\theta|\le m^\alpha} |\phi_m(\theta)|^n \,\mathrm{d}\theta.$$

We bound the integrand pointwise. Since $\hat{\pi}_m(\cdot)$ is invariant under rotations preserving $\mathbb{Z}^d$, it suffices to look at θ with all components positive. Assume $\theta_1 = \max\{\theta_1, \ldots, \theta_d\}$. Set $M = \lfloor 2\pi\sqrt{\gamma_m}/\theta_1 \rfloor$ and $K = \lfloor 5m/M \rfloor$. Notice that $\hat{\pi}_m(x) > 0$ implies $|x| < 2m$. By taking A large enough, we can assume that on the domain of integration, $M \leq m$. First,

$$\phi_m(\theta) = \sum_{(x_2,\ldots,x_d)} \exp\left(\frac{i}{\sqrt{\gamma_m}} \sum_{s=2}^{d} x_s\theta_s\right) \sum_{j=1}^{K} \sum_{x_1=-2m+(j-1)M}^{-2m+jM-1} \exp\left(\frac{ix_1\theta_1}{\sqrt{\gamma_m}}\right) \hat{\pi}_m(x).$$

Inside the x_1-summation, we write for each j separately

$$\hat{\pi}_m(x) = \hat{\pi}_m(x) - \hat{\pi}_m(x^{(j)}) + \hat{\pi}_m(x^{(j)}),$$

where $x^{(j)} = (-2m + (j-1)M, x_2, \ldots, x_d)$. By Corollary 10.1,

$$\left|\hat{\pi}_m(x) - \hat{\pi}_m(x^{(j)})\right| \leq C \left|\frac{x_1 + 2m - (j-1)M}{m}\right|^{1/2} m^{-d}.$$

Thus,

$$\left|\sum_{x_1=-2m+(j-1)M}^{-2m+jM-1} \exp\left(\frac{ix_1\theta_1}{\sqrt{\gamma_m}}\right) \left(\hat{\pi}_m(x) - \hat{\pi}_m(x^{(j)})\right)\right| \leq C\theta_1^{-3/2} m^{-d+1},$$

and

$$\left|\sum_{j=1}^{K} \sum_{x_1=-2m+(j-1)M}^{-2m+jM-1} \exp\left(\frac{ix_1\theta_1}{\sqrt{\gamma_m}}\right) \left(\hat{\pi}_m(x) - \hat{\pi}_m(x^{(j)})\right)\right| \leq C\theta_1^{-1/2} m^{-d+1}.$$

On our domain of integration, $0 < \left(\theta_1/\sqrt{\gamma_m}\right) \leq Cm^{\alpha-1} < 2\pi$ for large m. Therefore,

$$\begin{aligned}
\left|\sum_{j=1}^{K} \hat{\pi}_m(x^{(j)}) \sum_{x_1=-2m+(j-1)M}^{-2m+jM-1} \exp\left(\frac{ix_1\theta_1}{\sqrt{\gamma_m}}\right)\right| &\leq CKm^{-d} \left|\frac{1 - \exp\left(i\theta_1 M/\sqrt{\gamma_m}\right)}{1 - \exp\left(i\theta_1/\sqrt{\gamma_m}\right)}\right| \\
&\leq C|\theta| m^{-d},
\end{aligned}$$

and altogether for sufficiently large A, m and n,

$$\int\limits_{A<|\theta|\leq m^\alpha} |\phi_m(\theta)|^n \, \mathrm{d}\theta \leq C_1^n \int\limits_{A<|\theta|\leq m^\alpha} \left(\frac{1}{\sqrt{|\theta|}} + \frac{|\theta|}{m}\right)^n \mathrm{d}\theta = O\left(n^{-(d+2)/2}\right).$$

For $I_{3,3}$ we again assume all components of θ positive and $\theta_1 = \max\{\theta_1, \ldots, \theta_d\}$. Since

$$\hat{\pi}_m(x) = \sum_{y=-2m}^{x_1} \left(\hat{\pi}_m(y, x_2, \ldots, x_d) - \hat{\pi}_m(y-1, x_2, \ldots, x_d)\right),$$

we have

$$\begin{aligned}
&|\phi_m(\theta)| \\
&\le \; Cm^{d-1}\left|\sum_{x_1=-2m}^{2m}\exp\left(\frac{ix_1\theta_1}{\sqrt{\gamma_m}}\right)\sum_{y=-2m}^{x_1}\left(\hat\pi_m(y,x_2,\dots,x_d)-\hat\pi_m(y-1,x_2,\dots,x_d)\right)\right| \\
&\le \; Cm^{d-1}\sum_{y=-2m}^{2m}|\hat\pi_m(y,x_2,\dots,x_d)-\hat\pi_m(y-1,x_2,\dots,x_d)|\left|\sum_{x_1=y}^{2m}\exp\left(\frac{ix_1\theta_1}{\sqrt{\gamma_m}}\right)\right|.
\end{aligned}$$

The sum over the exponentials is estimated by $Cm/|\theta|$, so that again with Corollary 10.1,

$$|\phi_m(\theta)| \le C_2 m^{1/2}|\theta|^{-1}.$$

Hence, for α close to 1 and large n, m,

$$\int_{m^\alpha<|\theta|,\,\theta\in B_m} |\phi_m^n(\theta)|\,\mathrm{d}\theta \le C_2^n m^{n/2+\alpha(d-n)} = O\left(n^{-(d+2)/2}\right).$$

□

For Proposition 4.2, we still need a large deviation estimate.

Lemma 10.4 (Large deviation estimate)**.** *There exist constants $c_1, c_2 > 0$ such that for $|x| \ge 3m$*

$$\hat\pi_m^n(x) \le c_1 m^{-d}\exp\left(-\frac{|x|^2}{c_2 n m^2}\right).$$

Proof: Write P for $\mathrm{P}_{0,\hat\pi_m}$ and E for the expectation with respect to P, and denote by X_n^j the jth component of the random walk X_n under P. For $r > 0$,

$$\begin{aligned}
\sum_{y:|y|\ge r}\hat\pi_m^n(y) \;&\le\; \sum_{j=1}^{d}\mathrm{P}(|X_n^j| \ge d^{-1/2}r) \\
&=\; 2d\,\mathrm{P}(X_n^1 \ge d^{-1/2}r).
\end{aligned}$$

We claim that

$$\mathrm{P}(X_n^1 \ge d^{-1/2}r) \le \exp\left(-\frac{r^2}{8dnm^2}\right).$$

By the martingale maximal inequality for all $t, \lambda > 0$,

$$\mathrm{P}(X_n^1 \ge \lambda) \le e^{-t\lambda}\,\mathrm{E}\left[\exp(tX_n^1)\right] = e^{-t\lambda}\left(\mathrm{E}\left[\exp(tX_1^1)\right]\right)^n.$$

Since $X_1^1 \in (-2m, 2m)$ and $x \to e^{tx}$ is convex, it follows that

$$\exp(tX_1^1) \le \frac{1}{2}\frac{(2m-X_1^1)}{2m}e^{-2tm} + \frac{1}{2}\frac{(2m+X_1^1)}{2m}e^{2tm}.$$

Therefore, using the symmetry of X_1^1,

$$\mathrm{E}\left[\exp\left(tX_n^1\right)\right] \leq \left(\frac{1}{2}e^{-2tm} + \frac{1}{2}e^{2tm}\right)^n = \cosh^n(2tm) \leq e^{2nt^2m^2},$$

and

$$\mathrm{P}(X_n^1 \geq d^{-1/2}r) \leq e^{-td^{-1/2}r}e^{2nt^2m^2}.$$

Putting $t = r/(4\sqrt{d}\,nm^2)$ we get

$$\mathrm{P}(X_n^1 \geq d^{-1/2}r) \leq \exp\left(-\frac{r^2}{8dnm^2}\right).$$

From this it follows that

$$\begin{aligned} \hat{\pi}_m^n(x) = \sum_{y:|y|\geq|x|-2m} \hat{\pi}_m^{n-1}(y)\hat{\pi}_m(x-y) &\leq \frac{c_1}{m^d}\exp\left(-\frac{(|x|-2m)^2}{8d(n-1)m^2}\right) \\ &\leq \frac{c_1}{m^d}\exp\left(-\frac{|x|^2}{c_2nm^2}\right). \end{aligned}$$

□

Proof of Proposition 4.2: (i) follows from Proposition 4.1. For (ii), we set

$$N = N(x,m) = \frac{|x|^2}{\gamma_m}\left(\log\frac{|x|^2}{\gamma_m}\right)^{-2}.$$

We split $\hat{g}_{m,\mathbb{Z}^d}(x)$ into

$$\hat{g}_{m,\mathbb{Z}^d}(x) = \sum_{n=1}^{\infty}\hat{\pi}_m^n(x) = \sum_{n=1}^{\lfloor N\rfloor}\hat{\pi}_m^n(x) + \sum_{n=\lfloor N\rfloor+1}^{\infty}\hat{\pi}_m^n(x).$$

For the first sum on the right, we use the large deviation estimate from Lemma 10.4,

$$\sum_{n=1}^{\lfloor N\rfloor}\hat{\pi}_m^n(x) \leq c_1m^{-d}\sum_{n=1}^{\lfloor N\rfloor}\exp\left(-\frac{|x|^2}{c_2\,nm^2}\right) = O\left(|x|^{-d}\right).$$

In the second sum, we replace the transition probabilities by the expressions obtained in Proposition 4.1. The error terms are estimated by

$$\sum_{n=\lfloor N\rfloor+1}^{\infty} O\left(m^{-d}n^{-(d+2)/2}\right) = O\left(|x|^{-d}\left(\log\frac{|x|^2}{\gamma_m}\right)^d\right).$$

Putting $t_n = 2\gamma_m n/|x|^2$, we obtain for the main part

$$\begin{aligned} &\sum_{n=\lfloor N\rfloor+1}^{\infty}\frac{1}{(2\pi\gamma_m n)^{d/2}}\exp\left(-\frac{|x|^2}{2\gamma_m n}\right) \\ &= \frac{|x|^{-d+2}}{2\pi^{d/2}\gamma_m}\sum_{n=\lfloor N\rfloor+1}^{\infty} t_n^{-d/2}\exp(-1/t_n)(t_n - t_{n-1}) \\ &= \frac{|x|^{-d+2}}{2\pi^{d/2}\gamma_m}\int_0^{\infty} t^{-d/2}\exp(-1/t)\mathrm{d}t + O\left(|x|^{-d}\right). \end{aligned}$$

This proves the statement for $|x| \geq 3m$ with

$$c(d) = \frac{1}{2\pi^{d/2}} \int_0^\infty t^{-d/2} \exp(-1/t) \mathrm{d}t.$$

$\square$

Acknowledgments

I would like to thank Erwin Bolthausen and Ofer Zeitouni for many helpful discussions.

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Part 2: Coagulation and Fragmentation Processes

On a ternary coalescent process

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Abstract

We present a coalescent process where three particles merge at each coagulation step. Using a random walk representation, we prove duality with a fragmentation process, whose fragmentation law we specify explicitly. Furthermore, we give a second construction of the coalescent in terms of random binary forests and study asymptotic properties. Starting from N particles of unit mass, we obtain under an appropriate rescaling when N tends to infinity a well-known binary coalescent, the so-called standard additive coalescent.

Subject classifications: 60J25; 60J65.
Key words: coagulation, fragmentation, additive coalescent, random forest, Brownian excursion, ladder epochs.

0 Introduction

Generally speaking, a stochastic coalescent is a Markov process describing the coagulation of particles characterized by their size only. The rate at which particles merge depends just on the members involved. Conversely, fragmentation processes describe a Markovian evolution of particles which split independently into new particles (branching property). The goal of this paper is to study the stochastic coalescent with ternary coagulation kernel

$$\kappa(r,s,t) = r+s+t+3, \quad r,s,t>0,$$

to which we will simply refer to as ternary coalescent or ternary coalescent process. Here, three particles of sizes (masses) r,s,t coagulate into a new particle of size $r+s+t$ at rate $r+s+t+3$. Although at first glance, the kernel κ may look somewhat arbitrary (for example, it is not scale invariant), the corresponding process enjoys rather interesting properties. Similar to the additive coalescent, that is the coalescent where two particles with masses s, t merge at rate $\tilde{\kappa}(s,t) = s+t$, the state chain of the ternary coalescent admits different representations. In the spirit of Bertoin [5], we show how it can be obtained by looking at excursion intervals of a one-dimensional conditioned random walk. As a by-product of our representation, we establish duality with a fragmentation

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process via time-reversal. We stress that this is a unusual feature, because the branching property normally fails when time is reversed in a coalescent process. Section 7 of Bertoin [6] gives a brief overview over cases where such a duality relation has been proven. See also Chapter 5.5 in Pitman's lecture notes [19] for further discussions.

Using the same construction, we study asymptotic properties of the ternary coalescent starting from N particles of unit mass. Properly rescaled in space and time, we observe in the limit $N \to \infty$ the so-called standard additive coalescent, which has been obtained by Evans and Pitman in [12] as the weak limit $n \to \infty$ of the (binary) additive coalescent, started at time $-(1/2)\ln n$ with n atoms of size $1/n$. Here, Bertoin's characterization [4] of the dual fragmentation process connected to the standard additive coalescent by time-reversal plays a pivotal role. We emphasize that even though κ is a ternary coagulation kernel, we end up in the limit with a binary coagulation process.

We also highlight a second construction of the ternary coalescent involving random binary forests, following the ideas of Pitman in [18]. In a final remark, we point out that this representation could instead be used to work out our results. Moreover, we outline a possible extension of the results to certain k-ary coalescent processes.

The rest of this paper is organized as follows. In the first section, we describe the semigroup of the ternary coalescent and derive some further properties. We finish this part by computing the one-dimensional statistics for the underlying state chain starting from an odd number of particles of unit mass. Its special form already hints at a connection to hitting times of a one-dimensional nearest neighbor random walk, which we elaborate in the next section. There we prove duality via time-reversal with a fragmentation process, using an explicit construction of the coalescent in terms of ladder epochs. In the third part, we turn our attention to random binary trees and find a second interpretation of the ternary coalescent which is based on random binary forests. Finally we use again the random walk representation to study asymptotic properties of the coalescent in the last section.

1 Some basic properties

Throughout this text, let

$$\mathbb{N} = \{1, 2, \ldots\}, \quad \mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}, \quad \mathbb{Z}_+ = \mathbb{N} \cup \{0\}.$$

The coalescent process will take values in the space of decreasing numerical sequences with finitely many non-zero terms

$$\mathcal{S}^{\downarrow} = \{\mathbf{s} = (s_1, s_2, \ldots) : s_1 \geq s_2 \geq \ldots \geq 0, s_k = 0 \text{ for } k \text{ sufficiently large}\}.$$

We may think of elements of a sequence $\mathbf{s} \in \mathcal{S}^{\downarrow}$ as (sizes of) atoms or particles and simply identify $\mathbf{s}$ with its non-zero components. If we write $\mathbf{s} = (s_1, \ldots, s_l)$, the non-zero components of $\mathbf{s}$ are precisely given by $s_1, \ldots, s_l$. If $\mathbf{s} = (s_1, s_2, \ldots) \in \mathcal{S}^{\downarrow}$ and $1 \leq i < j < k$, we use the notation $\mathbf{s}^{i \oplus j \oplus k}$ for the sequence in $\mathcal{S}^{\downarrow}$ obtained from $\mathbf{s}$ by merging its ith, jth and kth terms, that is one removes s_i, s_j, s_k and rearranges the remaining elements together with the sum $s_i + s_j + s_k$ in decreasing order.

Let us define the object of our interest. Recall the kernel κ from the introduction.

Definition 1.1. The ternary coalescent with values in $\mathcal{S}^{\downarrow}$ and kernel κ is a continuous time Markov process $\mathcal{X} = (\mathcal{X}(t),\, t \geq 0)$ with state space $\mathcal{S}^{\downarrow}_{\prime}$ for an appropriate subset $\mathcal{S}^{\downarrow}_{\prime}$ of $\mathcal{S}^{\downarrow}$, and jump rates

$$q(\mathbf{s}, \cdot) = \sum_{1 \leq i < j < k,\, s_k > 0} \kappa(s_i, s_j, s_k)\delta_{\mathbf{s}^{i\oplus j\oplus k}}.$$

This definition can be adapted in an obvious way to other coagulation kernels, leading to different stochastic coalescent models, for example the additive coalescent with kernel $\tilde{\kappa}(s,t) = s + t$.

Before looking at concrete realizations, we collect in this section some basic properties which can be read off from the kernel κ and the very definition of jump-hold processes of the above type. Denote by $\mathcal{X} = (\mathcal{X}(t),\, t \geq 0)$ the ternary coalescent, started from a finite configuration $\mathbf{r} = (r_1, \ldots, r_N) \in \mathcal{S}^{\downarrow}$, where $N = 2n+1$, $n \in \mathbb{Z}_+$. We write $M = r_1 + \ldots + r_N$ for the total mass in the system. For every $k = 0, \ldots, n+1$, let T_k be the instant of the kth coagulation, with the convention $T_0 = 0$, $T_{n+1} = \infty$. The state chain or skeleton chain $\mathcal{X}'$ of the coalescent process is given by $\mathcal{X}'_k = \mathcal{X}(T_k)$, $k = 0, \ldots, n$. We use the expression $\#(t)$ for the number of particles at time t, whereas $J(t) = \max\{k \in \mathbb{Z}_+ : T_k \leq t\}$ stands for the number of jumps up to time t. Note that $\#(t) = N - 2J(t)$.

1.1 State chain and semigroup

Proposition 1.1. *In the preceding notation, the following holds true.*

(i) *The sequence $\Delta_k = T_k - T_{k-1}$, $k = 1, \ldots, n$, of the waiting times between two coagulations is a sequence of independent exponential variables with respective parameters*

$$\alpha(k) = \frac{1}{2}(M + N + 2 - 2k)(N + 1 - 2k)(N - 2k).$$

In particular, the sequences $\{T_k\}_{0 \leq k \leq n}$ and $\{\mathcal{X}'_k\}_{0 \leq k \leq n}$ are independent.

(ii) *The sequence $\{\mathcal{X}'_k\}_{0 \leq k \leq n}$ is a Markov chain with transition probabilities*

$$\mathbb{P}\left(\mathcal{X}'_{l+1} = \mathbf{s}^{i\oplus j\oplus k} \mid \mathcal{X}'_l = \mathbf{s}\right) = \frac{s_i + s_j + s_k + 3}{\alpha(l+1)},$$

where $0 \leq l < n$, $1 \leq i < j < k \leq N - 2l$, and $\mathbf{s} = (s_1, \ldots, s_{N-2l}) \in \mathcal{S}^{\downarrow}$ is a generic finite configuration with total mass $s_1 + \ldots + s_{N-2l} = M$ such that $\mathbb{P}(\mathcal{X}'_l = \mathbf{s}) > 0$.

Proof: Let $0 \leq l < n$, and put $L = N - 2l$. By construction, the time Δ_{l+1} between the lth and the $(l+1)$th coagulation given $\mathcal{X}'_l = \mathbf{s} = (s_1, \ldots, s_L)$ is exponentially distributed with parameter

$$\begin{aligned}
&\sum_{1\le i<j<k\le L} (s_i + s_j + s_k + 3) \\
&= 3\binom{L}{3} + \frac{1}{6}\left(\sum_{i,j,k=1}^{L}(s_i+s_j+s_k) - 3\sum_{i=1}^{L} s_i - 3\sum_{\substack{i,j=1,\\ i\neq j}}^{L}(2s_i+s_j)\right) \\
&= \frac{1}{2}(M+L)(L-1)(L-2) = \alpha(l+1).
\end{aligned}$$

Therefore, the waiting times $\{\Delta_k\}_{1\le k\le n}$ do not depend on the states $\{\mathcal{X}'_k\}_{1\le k\le n}$. The rest follows from the construction of our process. □

We turn to a description of the semigroup. Recall that $\mathcal{X}$ starts from $\mathcal{X}(0) = \mathbf{r} = (r_1, \dots, r_N)$. In the following, Γ denotes the Gamma function.

Proposition 1.2. *In the notation above, consider a partition π of $\{1,\dots,N\}$ into $N-2l$ (non-empty) blocks $B_1,\dots,B_{N-2l}$, each of odd cardinality. Denote by $\Lambda'_\pi(N-2l)$ the event that the $N-2l$ atoms of $\mathcal{X}'_l$ result from the coagulation of particles $\{r_i : i \in B_j\}$, $j = 1,\dots,N-2l$. Then, with $\mathbf{r}_{B_j} = \sum_{i\in B_j} r_i$,*

$$\mathbb{P}\left(\Lambda'_\pi(N-2l)\right) = \frac{l!}{\alpha(1)\cdots\alpha(l)} \prod_{j=1}^{N-2l} \frac{\Gamma\left((\mathbf{r}_{B_j}+|B_j|+2)/2\right)(|B_j|-1)!}{\Gamma\left((\mathbf{r}_{B_j}+3)/2\right)((|B_j|-1)/2)!}.$$

Proof: The first coagulation involves three particles with labels in the block B_j with probability

$$\sum_{i<i'<i''\in B_j} \frac{r_i + r_{i'} + r_{i''} + 3}{\alpha(1)} = \frac{(\mathbf{r}_{B_j}+|B_j|)(|B_j|-1)(|B_j|-2)}{2\alpha(1)}.$$

Now consider an arbitrary sequence $(k_1,\dots,k_l)$ taking values in $\{1,\dots,N-2l\}$ such that for every $j = 1,\dots,N-2l$, $|\{i \le l : k_i = j\}| = (|B_j|-1)/2$. Using the Markov property of $\mathcal{X}'$, we see that the probability that for all $i = 1,\dots,l$, the ith coagulation affected only particles formed from initial particles with labels in B_{k_i} equals

$$\frac{1}{\alpha(1)\cdots\alpha(l)} \prod_{j=1}^{N-2l} \frac{\Gamma\left((\mathbf{r}_{B_j}+|B_j|+2)/2\right)}{\Gamma\left((\mathbf{r}_{B_j}+3)/2\right)}(|B_j|-1)!\,.$$

Observe that the number of such sequences $(k_1,\dots,k_l)$ is

$$\binom{l}{(|B_1|-1)/2,\dots,(|B_{N-2l}|-1)/2} = \frac{l!}{((|B_1|-1)/2)!\cdots((|B_{N-2l}|-1)/2)!}.$$

This proves the statement. □

In the setting of the proposition, denote by $\Lambda_\pi(t)$ the event that $\mathcal{X}(t)$ has $N-2l$ atoms, each resulting from the merging of $\{r_i : i \in B_j\}$, $j=1,\dots,N-2l$. Since the sequence of coagulation times and the skeleton chain $\mathcal{X}'$ are independent,

$$\mathbb{P}(\Lambda_\pi(t)) = \mathbb{P}(T_l \le t < T_{l+1},\, \Lambda'_\pi(N-2l)) = \mathbb{P}(\#(t) = N-2l)\,\mathbb{P}(\Lambda'_\pi(N-2l)).$$

In particular, the semigroup of $\mathcal{X}$ is described by the preceding proposition and the distribution of the number of particles at time t, which is computed in the following lemma.

Lemma 1.1. *In the notation above, for $l = 0,\dots,n$ and $t \ge 0$,*

$$\mathbb{P}(\#(t) = N-2l) = \sum_{j=1}^{l+1} \frac{\alpha(j)e^{-\alpha(j)t}}{\alpha(l+1)} \prod_{k=1,k\neq j}^{l+1} \frac{\alpha(k)}{\alpha(k)-\alpha(j)}.$$

Proof: We use

$$\mathbb{P}(\#(t) = N-2l) = \mathbb{P}(T_{l+1} > t) - \mathbb{P}(T_l > t).$$

Note that T_k is distributed according to $\sum_{i=1}^k \alpha(i)^{-1}\mathbf{e}_i$, where $\alpha(i)$ is as in the statement of Proposition 1.1, and $\mathbf{e}_1, \mathbf{e}_2, \dots$ is a sequence of independent standard exponential variables. As a general fact, a sum of k independent exponential variables with pairwise distinct parameters $\alpha(i) > 0$ follows the hypoexponential distribution, that is the probability distribution with density

$$f(x) = \sum_{i=1}^k \alpha(i)e^{-\alpha(i)x} \prod_{j=1,j\neq i}^k \frac{\alpha(j)}{\alpha(j)-\alpha(i)}.$$

Integrating the density and regrouping terms result in the statement of the lemma. □

1.2 The monodisperse case

We turn to the situation where $\mathcal{X}(0) = \mathbf{r} = (1,\dots,1)$, that is the coalescent process is started from the monodisperse configuration consisting of $N = 2n+1$ atoms of unit mass. In this case, the total mass M equals N, so the rates $\alpha(i)$ simplify to

$$\alpha(i) = (N+1-i)(N+1-2i)(N-2i). \tag{1}$$

If $\mathbf{s} = (s_1,\dots,s_m) \in \mathcal{S}^\downarrow$ we denote by $\gamma(\mathbf{s})$ the number of different m-tuples that can be built from the elements s_i (recall that by our convention $s_i > 0$). To put it into a formula, if $\{s_{l_i}\}_{1\le i\le p}$ is a maximal family of pairwise disjoint non-zero elements from the sequence $\mathbf{s}$, and $k_i = |\{j=1,\dots,m : s_j = s_{l_i}\}|$, we define

$$\gamma(\mathbf{s}) = \binom{m}{k_1,\dots,k_p}.$$

In other words, the ranking map

$$rk : \bigcup_{m=1}^{\infty} \mathbb{N}^m \longrightarrow \mathcal{S}^{\downarrow}$$

which orders $(r_1, \ldots, r_m) \in \mathbb{N}^m$ decreasingly satisfies $|rk^{-1}(\mathbf{s})| = \gamma(\mathbf{s})$ for each $\mathbf{s} \neq (0, \ldots) \in \mathcal{S}^{\downarrow}$. As a corollary of Proposition 1.2, the one-dimensional statistics for $\mathcal{X}'$ look as follows.

Corollary 1.1. *Let $0 \leq l \leq n$ and $\mathbf{s} = (s_1, \ldots, s_{N-2l}) \in \mathcal{S}^{\downarrow}$ with $s_i \in \mathbb{N}$ odd for all i, and $s_1 + \ldots + s_{N-2l} = N$. Then, in the situation described above,*

$$\mathbb{P}\left(\mathcal{X}'_l = \mathbf{s}\right) = \gamma(\mathbf{s}) \frac{N}{N-2l} \binom{N}{l}^{-1} \prod_{i=1}^{N-2l} \frac{1}{s_i} \binom{s_i}{\frac{s_i+1}{2}}.$$

Proof: The starting configuration is given by $(r_1, \ldots, r_N)$ with $r_i = 1$ for each i. Thus, if $\mathcal{X}'_l$ has $N-2l$ atoms of the sizes $s_1 \geq \ldots \geq s_{N-2l}$, then there is a partition π of $\{1, \ldots, N\}$ into $N-2l$ blocks $B_1, \ldots, B_{N-2l}$ of cardinality $|B_j| = s_j$, such that the atoms of $\mathcal{X}'_l$ evolved from merging the particles $\{r_i : i \in B_j\}$. Denote this event by $\Lambda'_\pi(N-2l)$. Since

$$\alpha(1) \cdots \alpha(l) = \frac{N!(N-1)!}{(N-l)!(N-2l-1)!},$$

we obtain from Proposition 1.2 (note that here $\mathbf{r}_{B_j} = |B_j| = s_j$)

$$\mathbb{P}\left(\Lambda'_\pi(N-2l)\right) = \frac{(N-1-2l)!}{(N-1)!} \binom{N}{l}^{-1} \prod_{i=1}^{N-2l} (s_i - 1)! \binom{s_i}{\frac{s_i+1}{2}}.$$

The number of such partitions π is given by

$$\frac{\gamma(\mathbf{s})}{(N-2l)!} \binom{N}{s_1, \ldots, s_{N-2l}}.$$

By multiplying the last two expressions together, we arrive at the stated expression. □

As the reader may already check at this stage, $\mathcal{X}'_l$ has the same distribution as the decreasingly ranked sequence of $N-2l$ independent copies ξ_i of the first hitting time of -1 of a simple random walk, conditioned on $\xi_1 + \ldots + \xi_{N-2l} = N$ (see Section 2.3 for a definition of these quantities). Indeed, if $\xi_{(k)}$ denotes the kth order statistic of $\xi_1, \ldots, \xi_{N-2l}$, then for $\mathbf{s} = (s_1, \ldots, s_{N-2l}) \in \mathcal{S}^{\downarrow}$

$$\begin{aligned} &\mathbb{P}\left((\xi_{(N-2l)}, \ldots, \xi_{(1)}) = (s_1, \ldots, s_{N-2l}) \,|\, \xi_1 + \ldots + \xi_{N-2l} = N \right) \\ &\quad = \gamma(\mathbf{s})\, \mathbb{P}\left((\xi_1, \ldots, \xi_{N-2l}) = (s_1, \ldots, s_{N-2l}) \,|\, \xi_1 + \ldots + \xi_{N-2l} = N \right), \end{aligned}$$

and an application of Lemma 2.1 affirms that the last expression coincides with that obtained in the corollary. The connection between random walks and the ternary coalescent will become much clearer in the next section.

2 Duality with fragmentation via random walks

Our intention of this section is to prove duality of the ternary coalescent with a fragmentation process. Let us begin with an informal description of such processes.

Conversely to the phenomenon of coagulation of particles, one often observes in nature or science processes of fragmentation. In these systems, particles are broken into smaller pieces as time passes. As an example, one may think of DNA fragmentation in biology or fractures in geophysics. Just as for coalescent processes, one needs to impose constraints on such systems to make them mathematically tractable. First, one assumes that the process has no memory in the sense that the future does only depend on the present state and not on the past. Second, one supposes that a particle is entirely characterized by its size, that is by a real number, and third, one requires the system to fulfill the branching property, which means that particles split independently.

Naively, one might first guess that a coalescent process can always be turned into a fragmentation process by reversing time. However, even though the memoryless property is preserved under time reversal, the branching property is typically not fulfilled. In fact, there are only few examples known where a duality relation holds (see [6] Section 7 for an overview).

In view of our informal characterization, it is natural to call a Markov process with values in $\mathcal{S}^{\downarrow}$ a *ternary fragmentation process*, if each particle splits at a certain rate according to some dislocation law into three smaller pieces, where both the rate and the dislocation law depend only on the particle size s, and the sizes of the newly formed elements sum up to s. Ranked in decreasing order, these three particles together with the ones that did not split form the next state of the process. In particular, different particles split independently.

For our ternary coalescent starting from $N = 2n + 1$ atoms of unit mass, we shall prove

Theorem 2.1. *Reversing the coalescent chain $\{\mathcal{X}'_k\}_{0\leq k\leq n}$ in time results in the state chain of the fragmentation process, whose dynamics are given in Proposition 2.2.*

We will derive our result from an explicit construction of the skeleton chain $\mathcal{X}'$ in terms of (lengths of) excursion intervals of a conditioned random walk. This representation will also be useful for studying asymptotic properties in the last section.

2.1 From configurations to paths to mass partitions

We first show how subsets of $\{0, 1, \ldots, 2n\}$ can be identified with certain paths of nearest neighbor walks on $\mathbb{Z}$ of length $2n + 1$. The excursion intervals above two consecutive (new) minima of such paths partition the space $\mathbb{Z}/(2n + 1)\mathbb{Z}$ into discrete arcs. Taking the ranked sequence of their lengths, we obtain the main object of our interest.

To begin with, define the configuration space $\mathcal{C}_n$ to be the set of all subsets of $\{0, \ldots, 2n\}$ which have cardinality less or equal to n. We often represent $x \in \mathcal{C}_n$ by the vector $(x(i))_{0\le i\le 2n}$, where

$$x(i)=\begin{cases}1\ , & i\in x\\ 0\ , & i\notin x\end{cases}.$$

Under this identification, we may regard x as a mass distribution. We use the terminology that a site i is occupied by a mass if $x(i)=1$ and vacant otherwise. The number of occupied sites (the cardinality of the subset x) is denoted by

$$|x|=|\{i\in\{0,\ldots,2n\}:x(i)=1\}|\,.$$

We identify a configuration $x\in\mathcal{C}_n$ with a path of a nearest neighbor walk of length $2n+1$ on $\mathbb{Z}$ in the following way. Starting from the origin at time zero, the walk goes one step up if site 0 is occupied, i.e. $x(0)=1$, and down otherwise, then above if $x(1)=1$, down if $x(1)=0$ and so on, up to time $2n$. More precisely, the corresponding path $S(x)$ is given by $S(x)_0=0$ and for $1\le j\le 2n+1$,

$$S(x)_j=2\left(\sum_{i=0}^{j-1}x(i)\right)-j.$$

Notice that by definition, $S(x)_{2n+1}=2(|x|-n)-1$. Clearly, the mapping $\mathcal{C}_n\ni x\mapsto S(x)$ is one-to-one.

As we show next, the excursion intervals of such a path provide us with an element $\varphi_1(x)$ in the space of cyclically ordered partitions of $\mathbb{Z}/(2n+1)\mathbb{Z}$ into discrete arcs,

$$\begin{aligned}\mathcal{P}^{\circ}_{2n+1} &= \{\mathtt{s}^{\circ}=(\mathtt{s}_1,\ldots,\mathtt{s}_m): \text{there exist } a_1<a_2<\ldots<a_m\le 2n+1,\\ &\quad m,a_i\in\mathbb{N},\ \text{such that for } 1\le i\le m-1, \mathtt{s}_i=[a_i,a_{i+1})\cap\mathbb{N},\\ &\quad \mathtt{s}_m=([a_m,2n+1)\cup[0,a_1))\cap\mathbb{Z}_+\}\,.\end{aligned}$$

Take $x\in\mathcal{C}_n$, and let $M=-S(x)_{2n+1}$. With $\underline{m}(x)=\min_{0\le j\le 2n+1}S_j(x)$, define the first time at which $S(x)$ reaches $\underline{m}(x)+k$, $k=0,\ldots,M-1$,

$$m_k(S(x))=\inf\{j\ge 0:S_j(x)=\underline{m}(x)+k\}\,.$$

For $i=1,\ldots,M$, put $a_i=m_{M-i}(S(x))$. We construct a sequence $\mathtt{s}^{\circ}=(\mathtt{s}_1,\ldots,\mathtt{s}_M)\in\mathcal{P}^{\circ}_{2n+1}$ by setting $\mathtt{s}_i=[a_i,a_{i+1})\cap\mathbb{N}$ for $i=1,\ldots,M-1$, $\mathtt{s}_M=([a_M,2n+1)\cup[0,a_1))\cap\mathbb{Z}_+$. In other words, if we look for $k=0,\ldots,2n$ at the shifted path $\theta_k(S(x))$ defined by

$$\theta_k(S(x))_i=\begin{cases}S(x)_{i+k}-S(x)_k & ,\ 0\le i\le 2n+1-k\\ S(x)_{i+k-(2n+1)}+S(x)_{2n+1}-S(x)_k\ , & 2n+1-k<i\le 2n+1\end{cases},$$

then the element $\mathtt{s}^{\circ}$ corresponds to the M successive excursion intervals of $\theta_{m_{M-1}}S(x)$ above two consecutive (new) minima. The length $|\mathtt{s}_i|$ of such an interval is also referred to as a *ladder epoch*. We let $\varphi_1(x)=\mathtt{s}^{\circ}$ and define φ_2 as the function which sends

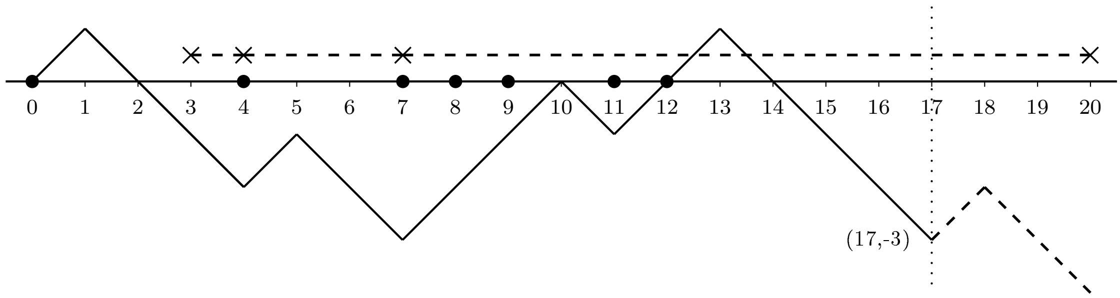

Figure 1: The black dots represent the configuration $x = \{0, 4, 7, 8, 9, 11, 12\} \subset \mathcal{C}_8$. The corresponding path $S(x)$ starts at zero and ends in -3 at time 17. It is periodically extended up to time 20 to better recognize the excursion intervals $\varphi_1(x)$. They are visualized by the dashed line above the x-axis, where the crosses mark the endpoints of the intervals, i.e. $\varphi_1(x) = (\mathtt{s}_1, \mathtt{s}_2, \mathtt{s}_3)$ with $\mathtt{s}_1 = [3, 4) \cap \mathbb{N}$, $\mathtt{s}_2 = [4, 7) \cap \mathbb{N}$, $\mathtt{s}_3 = ([7, 17) \cup [0, 3)) \cap \mathbb{Z}_+$.

$\mathtt{s}^\circ = (\mathtt{s}_1, \ldots, \mathtt{s}_m) \in \mathcal{P}^\circ_{2n+1}$ to its arc lengths $\{|\mathtt{s}_i|\}_{1 \le i \le m}$, arranged in decreasing order. In this way, we obtain an element in the space of mass partitions

$$\mathcal{P}^\downarrow_{2n+1} = \left\{ \mathbf{s} = (s_1, \ldots, s_m) : s_1 \ge s_2 \ge \ldots \ge s_m \, , m, s_i \in \mathbb{N}, \sum_{i=1}^m s_i = 2n+1 \right\}.$$

By filling up with an infinite sequence of zeros, we will often identify mass partitions with elements in $\mathcal{S}^\downarrow$. To summarize our construction, the concatenation map φ

$$\varphi = \varphi_2 \circ \varphi_1 : \mathcal{C}_n \xrightarrow{\varphi_1} \mathcal{P}^\circ_{2n+1} \xrightarrow{\varphi_2} \mathcal{P}^\downarrow_{2n+1} \subset \mathcal{S}^\downarrow.$$

sends configurations $x \in \mathcal{C}_n$ via their path representations to partitions of $\mathbb{Z}/(2n+1)\mathbb{Z}$ and then to mass partitions.

2.2 Random evolution

Our purpose here is to randomize the input of the map $\varphi : \mathcal{C}_n \to \mathcal{P}^\downarrow_{2n+1}$ to obtain (a sequence of) random mass partitions. More precisely, we construct two Markov chains on $\mathcal{C}_n$ running from time zero up to n as follows. Let $X = \{X_k\}_{0 \le k \le n}$ be the Markov chain with $X_0 = \emptyset$ and transition probabilities

$$p_X(x, y) = \begin{cases} \frac{1}{2n+1-|x|} & , \; x \subset y \text{ and } y \backslash x = \{i\} \text{ for some } i \in \{0, \ldots, 2n\} \backslash x \\ 0 & , \; \text{otherwise} \end{cases}.$$

In words, $(X_0, \ldots, X_l)$ is obtained by occupying successively l sites from $\{0, \ldots, 2n\}$, chosen uniformly at random. From the point of view of sets, X_l is uniformly distributed on the space of all l-subsets of $\{0, \ldots, 2n\}$. By identifying with the random path $S(X_l)$, we will also think of X_l as simple random walk up to time $2n+1$, conditioned to end at position $-2(n-l) - 1$.

Let $Y = \{Y_k\}_{0\le k\le n}$ be the Markov chain with Y_0 being uniformly distributed on the space of all n-subsets of $\{0,\ldots,2n\}$ and transition probabilities

$$p_Y(x,y) = \begin{cases} \frac{1}{|x|} & , \; y\subset x \text{ and } x\backslash y = \{i\} \text{ for some } i\in\{0,\ldots,2n\}\backslash y \\ 0 & , \; \text{otherwise} \end{cases} .$$

In words, $(Y_0,\ldots,Y_l)$ is obtained by removing successively l masses chosen uniformly at random from the starting configuration Y_0. In terms of sets, Y_l is uniformly distributed on the space of all $(n-l)$- subsets of $\{0,\ldots,2n\}$. As above, Y_l can be identified with simple random walk up to time $2n+1$, conditioned to end at $-2l-1$. Note that by construction, we have the duality relation

$$(X_0,\ldots,X_n) \stackrel{\mathrm{d}}{=} (Y_n,\ldots,Y_0). \tag{2}$$

2.3 Realization of the skeleton chains

We are not interested in X and Y themselves, but rather in $\varphi(X) = \{\varphi(X_k)\}_{0\le k\le n}$ and $\varphi(Y)$. As we will show in Proposition 2.3, the former is the state chain of the ternary coalescent starting from $N = 2n+1$ atoms of unit mass. The latter is characterized by Proposition 2.2 as the state chain of a fragmentation process starting from a single particle of mass N.

We need some preparation. Recall that simple random walk on $\mathbb{Z}$ is the Markov chain $S = \{S_m\}_{m\ge 0}$ with $S_0 = 0$ and $S_m = \zeta_1 + \ldots + \zeta_m$, where $\zeta_1, \zeta_2, \ldots$ are independent random variables with $\mathbb{P}(\zeta_i = \pm 1) = 1/2$. For $k\in\mathbb{Z}$, the first hitting time of k is denoted by

$$H_k = \inf\{m\ge 1 : S_m = k\}.$$

The following result on the distribution of H_k is classical.

Lemma 2.1. *Let $k\in\mathbb{Z}$, $k\ne 0$, and $m\in\mathbb{N}$. Then*

$$\mathbb{P}(H_k = m) = \begin{cases} \frac{|k|}{m}\binom{m}{(m+|k|)/2}2^{-m} & , \; k = m[\mathrm{mod}\ 2] \\ 0 & , \; k\ne m[\mathrm{mod}\ 2] \end{cases} .$$

Moreover, if $m = 2n+1$ and k is a fixed odd number, as $n\to\infty$,

$$\mathbb{P}(H_k = m) \sim \frac{1}{2}\sqrt{\frac{1}{\pi n^3}}.$$

Proof: Clearly, for the probability to be different from zero the numbers k and m must have the same parity. Then, using the hitting time theorem (see for example [14]) in the first equality,

$$\mathbb{P}(H_k = m) = \frac{|k|}{m}\mathbb{P}(S_m = k) = \frac{|k|}{m}\binom{m}{(m+|k|)/2}2^{-m}.$$

The second statement follows from Stirling's formula for the factorial. □

Before looking at $\varphi(X)$ and $\varphi(Y)$ in detail, let us give an indication that the former is the skeleton chain of the ternary coalescent. Recall Corollary 1.1 and the connection between $\mathcal{X}'$ and hitting times. Let $N = 2n+1$, $0 \le l \le n$, and take N independent copies ξ_i of the hitting time H_{-1}. Denote by $\xi_{(k)}$ the kth order statistic of $\xi_1, \ldots, \xi_{N-2l}$.

Proposition 2.1. *$\varphi(X_l)$ is distributed according to $(\xi_{(N-2l)}, \ldots, \xi_{(1)})$ conditionally on $\xi_1 + \ldots + \xi_{N-2l} = N$, i.e. the one-dimensional distributions of $\varphi(X)$ and $\mathcal{X}'$ started from N atoms of mass one agree.*

Proof: We identify X_l with simple random walk $S(X_l)$ up to time N, conditioned to end at $-(N-2l)$. For notational simplicity, let us write S instead of $S(X_l)$. Also recall the definitions of $\theta_k(S)$ and $m_k(S)$ from Section 2.1. By Theorem 1 of [8], if ν is a uniform random variable on $\{0, \ldots, N-2l-1\}$ independent of S, then the chain $\theta_{m_\nu}(S)$ has the law of S conditioned on $H_{-(N-2l)} = N$. Moreover, the index m_ν is uniformly distributed on $\{0, \ldots, N-1\}$ and independent of the chain $\theta_{m_\nu}(S)$. Denote by $\theta_k X_l$ the shifted configuration defined by $\theta_k X_l(i) = X_l(i + k[\mathrm{mod}\ N])$. Clearly, $\varphi(X_l) = \varphi(\theta_k X_l)$ for each k. From Theorem 1 of [8] we thus infer that for $(s_1, \ldots, s_{N-2l}) \in \mathcal{S}^{\downarrow}$,

$$\begin{aligned} &\mathbb{P}\left(\varphi(X_l) = (s_1, \ldots, s_{N-2l})\right) = \mathbb{P}\left(\varphi(\theta_{m_\nu} X_l) = (s_1, \ldots, s_{N-2l})\right) \\ &= \ \mathbb{P}\left((\xi_{(N-2l)}, \ldots, \xi_{(1)}) = (s_1, \ldots, s_{N-2l}) \,|\, \xi_1 + \ldots + \xi_{N-2l} = N\right). \end{aligned}$$

$\square$

For the moment, we leave $\varphi(X)$ aside and first turn to $\varphi(Y)$. In the sequel it is convenient to use the notion of multisets, which we distinguish from normal sets by using double braces. For example, $\{\{a, b, c, c\}\}$ contains the elements a, b each with multiplicity 1 and the element c with multiplicity 2. The cardinality of this multiset is 4, the order of elements is irrelevant, as for sets.

Let ξ_1, ξ_2, ξ_3 be three independent copies of the hitting time H_{-1}. To state the transition mechanism of $\varphi(Y)$ in a concise way, we define a family $\mu = (\mu_s, s \ge 3 \text{ odd})$ of probability laws, supported on

$$\Omega_s = \{R = \{\{r_1, r_2, r_3\}\} : r_i \in \mathbb{N} \text{ odd}, \, r_1 + r_2 + r_3 = s\},$$

by setting

$$\mu_s(R) = \mathbb{P}\left(\{\{\xi_1, \xi_2, \xi_3\}\} = R \mid \xi_1 + \xi_2 + \xi_3 = s\right). \tag{3}$$

More explicitly, applying Lemma 2.1 results in the expression

$$\mu_s(R) = \gamma \frac{s}{3 r_1 r_2 r_3} \binom{r_1}{\frac{r_1+1}{2}} \binom{r_2}{\frac{r_2+1}{2}} \binom{r_3}{\frac{r_3+1}{2}} \left[\binom{s}{\frac{s+3}{2}}\right]^{-1}, \tag{4}$$

where γ is the number of triplets (r_i, r_j, r_k) that can be formed from $R = \{\{r_1, r_2, r_3\}\}$,

$$\gamma = \begin{cases} 6\,, & |\{r_1, r_2, r_3\}| = 3 \\ 3\,, & |\{r_1, r_2, r_3\}| = 2 \\ 1\,, & |\{r_1, r_2, r_3\}| = 1 \end{cases}.$$

Proposition 2.2. $\varphi(Y) = \{\varphi(Y_k)\}_{0\le k\le n}$ *is a Markov chain. Its transition mechanism from time* $l \le n-1$ *to* $l+1$ *is described as follows.*

(a) Conditionally on $\varphi(Y_l) = \mathbf{s} = (s_1, \dots, s_{2l+1}) \in \mathcal{P}^{\downarrow}_{2n+1}$*, select an index* $\iota \in \{1, \dots, 2l+1\}$ *according to the law*

$$\mathbb{P}\left(\iota = i \mid \varphi(Y_l) = \mathbf{s}\right) = \frac{s_i - 1}{2(n-l)}.$$

(b) Given $\varphi(Y_l) = \mathbf{s}$ *and* $\iota = i$*, split* s_i *according to the law* μ_{s_i} *into three numbers and rank them together with* s_m*,* $m \in \{1, \dots, 2l+1\}\backslash\{i\}$*, in decreasing order to obtain a new mass partition.*

Proof: Fix $l \in \{0, \dots, n-1\}$. We write $\varphi(Y)_{0:i}$ for the vector $(\varphi(Y_0), \dots, \varphi(Y_i))$. The Markov property will follow from

(i) $\varphi(Y)_{0:l}$ and $\varphi(Y_{l+1})$ are conditionally independent given $\varphi_1(Y_l)$.

(ii) $\varphi_1(Y_l)$ and $\varphi(Y_{l+1})$ are conditionally independent given $\varphi(Y_l)$.

Indeed, assuming (i) and (ii), we have for $\mathbf{r}_{0:l+1} = (\mathbf{r}_0, \dots, \mathbf{r}_{l+1}) \in \mathcal{P}^{\downarrow}_{2n+1} \times \dots \times \mathcal{P}^{\downarrow}_{2n+1}$,

$$\begin{aligned}
&\mathbb{P}\left(\varphi(Y)_{0:l+1} = \mathbf{r}_{0:l+1}\right) \\
&= \sum_{\mathbf{u}:\varphi_2(\mathbf{u}^\circ)=\mathbf{r}_l} \mathbb{P}\left(\varphi(Y)_{0:l} = \mathbf{r}_{0:l} \mid \varphi_1(Y_l) = \mathbf{u}^\circ\right) \mathbb{P}\left(\varphi(Y_{l+1}) = \mathbf{r}_{l+1} \mid \varphi_1(Y_l) = \mathbf{u}^\circ\right) \\
&\qquad \times \mathbb{P}\left(\varphi_1(Y_l) = \mathbf{u}^\circ\right) \\
&= \mathbb{P}\left(\varphi(Y_{l+1}) = \mathbf{r}_{l+1} \mid \varphi(Y_l) = \mathbf{r}_l\right) \mathbb{P}\left(\varphi(Y)_{0:l} = \mathbf{r}_{0:l}\right).
\end{aligned}$$

For (i), the key step is to show that the conditional law of $\varphi(Y)_{0:l}$ given Y_l only depends on $\varphi_1(Y_l)$. In that direction, we work conditionally on $\varphi_1(Y_l) = \mathbf{s}^\circ = (\mathbf{s}_1, \dots, \mathbf{s}_{2l+1})$ and denote by $N_k(i) = |Y_k \cap \mathbf{s}_i|$ the number of sites of the arc $\mathbf{s}_i$ which are occupied by Y_k. Write N_k for the family $\{N_k(i)\}_{1\le i\le 2l+1}$. Let i_l denote the unique index such that the singleton $Y_{l-1}\backslash Y_l \subset \mathbf{s}_{i_l}$. In other words, i_l is the unique index i such that $N_{l-1}(i) = N_l(i) + 1$. Then $\varphi_1(Y_{l-1})$ results from $\varphi_1(Y_l) = \mathbf{s}$ by merging the arcs $\mathbf{s}_{i_l}$, $\mathbf{s}_{i_l+1}$ and $\mathbf{s}_{i_l+2}$ (with the convention that indices of arcs are taken modulo $2l+1$). By iteration, we realize that the sequence $N_{0:l} = (N_0, \dots, N_l)$ determines $\varphi_1(Y)_{0:l}$ and therefore also $\varphi(Y)_{0:l}$. Hence it now suffices to check that the conditional distribution of $N_{0:l}$ given Y_l only depends on $\varphi_1(Y_l) = \mathbf{s}^\circ$, which is straightforward from the dynamics and the observation that for every $i = 1, \dots, 2l+1$, the arc $\mathbf{s}_i$ has exactly $(|\mathbf{s}_i| + 1)/2$ sites which are not occupied by Y_l.

We are now able to prove (i). Take $\mathbf{t} \in \mathcal{P}^{\downarrow}_{2n+1}$ with $\mathbb{P}\left(\varphi_1(Y_l) = \mathbf{s}^\circ, \varphi(Y_{l+1}) = \mathbf{t}\right) > 0$. Then

$$\begin{aligned}
&\mathbb{P}\left(\varphi(Y)_{0:l} = \mathbf{r}_{0:l}, \varphi_1(Y_l) = \mathbf{s}^\circ, \varphi(Y_{l+1}) = \mathbf{t}\right) \\
&= \sum_{\substack{x\in\varphi_1^{-1}(\mathbf{s}^\circ),\\ y\in\varphi^{-1}(\mathbf{t})}} \mathbb{P}\left(\varphi(Y)_{0:l} = \mathbf{r}_{0:l} \mid Y_l = x, Y_{l+1} = y\right) \mathbb{P}\left(Y_l = x, Y_{l+1} = y\right).
\end{aligned}$$

Since Y is a Markov chain, it follows that for $x, y \in \mathcal{C}_n$ with $\mathbb{P}(Y_l = x, Y_{l+1} = y) > 0$,

$$\mathbb{P}\left(\varphi(Y)_{0:l} = \mathbf{r}_{0:l} \mid Y_l = x, Y_{l+1} = y\right) = \mathbb{P}\left(\varphi(Y)_{0:l} = \mathbf{r}_{0:l} \mid Y_l = x\right).$$

Plugging this into the above formula and using the conditional independence of $\varphi(Y)_{0:l}$ and Y_l given $\varphi_1(Y_l)$, we deduce that for $x \in \varphi_1^{-1}(\mathbf{s}^\circ)$,

$$\mathbb{P}\left(\varphi(Y)_{0:l} = \mathbf{r}_{0:l} \mid \varphi_1(Y_l) = \mathbf{s}^\circ, \varphi(Y_{l+1}) = \mathbf{t}\right) = \mathbb{P}\left(\varphi(Y)_{0:l} = \mathbf{r}_{0:l} \mid Y_l = x\right).$$

Similarly, one sees that the right hand side equals $\mathbb{P}(\varphi(Y)_{0:l} = \mathbf{r}_{0:l} \mid \varphi_1(Y_l) = \mathbf{s}^\circ)$, and (i) follows. We turn to (ii) and the description of the transition mechanism. We keep the conditioning on $\varphi_1(Y_l) = \mathbf{s}^\circ$. Note that Y_{l+1} evolves from Y_l by removing uniformly at random one of the $n-l$ masses. By identifying Y_l with $S(Y_l)$, this amounts to switching one of the upward steps chosen uniformly at random into a downward step. More precisely, under our conditioning, the ith arc $\mathtt{s}_i$ is picked with probability

$$\frac{\text{number of upward steps over } \mathtt{s}_i}{\text{total number of upward steps}} = \frac{(|\mathtt{s}_i| - 1)/2}{n-l}, \tag{5}$$

then one of the upward steps over $\mathtt{s}_i$ is selected with uniform probability and changed into a downward step. Up to a vertical shift in space, $S(Y_l)$ restricted to the arc $\mathtt{s}_i$ obeys the law of simple random walk conditioned on $H_{-1} = |\mathtt{s}_\mathtt{i}|$ (with an obvious modification for the last arc $\mathtt{s}_{2l+1}$). Given an upward step over $\mathtt{s}_\mathtt{i}$ is switched, $S(Y_{l+1})$ restricted to $\mathtt{s}_i$ can therefore be seen as simple random walk conditioned on $H_{-3} = |\mathtt{s}_\mathtt{i}|$. In terms of $\varphi(Y)$, we deduce that $\varphi(Y_{l+1})$ is obtained by first picking the ith arc $\mathtt{s}_i$ with probability given in (5), then splitting its length according to $\mu_{|\mathtt{s}_\mathtt{i}|}$ into three numbers r_1, r_2, r_3 corresponding to the first three ladder epochs of simple random walk conditioned on $H_{-3} = |\mathtt{s}_i|$, and finally ranking them together with the numbers $|\mathtt{s}_j|$, $j \neq i$, in decreasing order. In particular, we realize that for predicting $\varphi(Y_{l+1})$ out of $\varphi_1(Y_l)$, the additional information given by $\varphi_1(Y_l)$ compared to $\varphi(Y_l)$, namely the location of the arcs, is irrelevant. Hence also (ii) holds. □

Let us now characterize $\varphi(X)$.

Proposition 2.3. $\varphi(X) = \{\varphi(X_k)\}_{0 \leq k \leq n}$ *is a Markov chain. Its transition mechanism from time* $l \leq n-1$ *to* $l+1$ *is described as follows.*

(a) Conditionally on $\varphi(X_l) = \mathbf{s} = (s_1, \ldots, s_{2(n-l)+1}) \in \mathcal{P}^{\downarrow}_{2n+1}$, *select an index* ι *out of the set of all* 3*-subsets of* $\{1, \ldots, 2(n-l)+1\}$ *according to the law*

$$\mathbb{P}\left(\iota = \{i,j,k\} \mid \varphi(X_l) = \mathbf{s}\right) = \frac{s_i + s_j + s_k + 3}{(2n+1-l)2(n-l)(2(n-l)-1)}.$$

(b) Given $\varphi(X_l) = \mathbf{s}$ *and* $\iota = \{i,j,k\}$, *rank the sum* $r = s_i + s_j + s_k$ *together with the numbers* s_m, $m \in \{1, \ldots, 2(n-l)+1\} \backslash \{i,j,k\}$, *in decreasing order to obtain a new mass partition.*

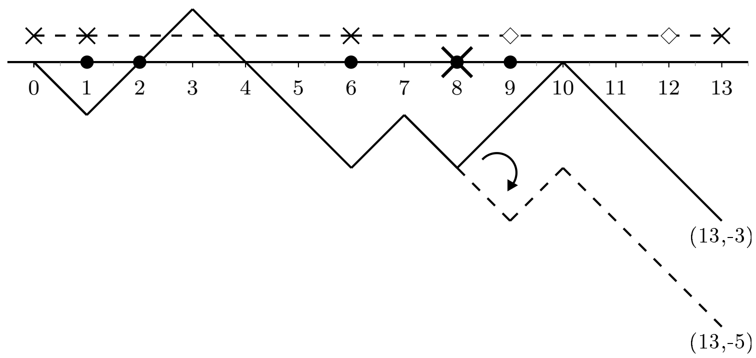

Figure 2: The transition mechanism from $\varphi(Y_1)$ to $\varphi(Y_2)$, where $n = 6$. Here, at time 1 the chain Y is in the configuration state $Y_1 = \{1, 2, 6, 8, 9\}$. Then the mass at position 8 is removed. For the corresponding path, this means that the upward step at time 8 is changed into a downward step. The new path $S(Y_2)$ coincides up to time 8 with the old path $S(Y_1)$ and is then indicated by the dashed line. The excursion interval $[6, 13)$ is broken into three intervals $[6, 9)$, $[9, 12)$, $[12, 13)$. Therefore, $\varphi_1(Y_2) = (\mathtt{s}_1, \mathtt{s}_2, \mathtt{s}_3, \mathtt{s}_4, \mathtt{s}_5)$ with $\mathtt{s}_1 = [1, 6) \cap \mathbb{N}$, $\mathtt{s}_2 = [6, 9) \cap \mathbb{N}$, $\mathtt{s}_3 = [9, 12) \cap \mathbb{N}$, $\mathtt{s}_4 = [12, 13) \cap \mathbb{N}$, $\mathtt{s}_5 = [0, 1) \cap \mathbb{Z}_+$ and $\varphi(Y_2) = (5, 3, 3, 1, 1)$.

Proof: From the duality (2) it follows that $\varphi(X)$ is obtained by reversing $\varphi(Y)$ in time. In particular, the Markov property carries over from $\varphi(Y)$ to $\varphi(X)$.

It remains to look at the transition mechanism. The step from $l = n-1$ to n is obvious from the construction of X and φ. Now fix $l \in \{0, \ldots, n-2\}$, and let $M = 2(n-l)+1$. We work conditionally on $\varphi(X_l) = \mathbf{s} = (s_1, \ldots, s_M) \in \mathcal{P}^{\downarrow}_{2n+1}$. By construction, $\varphi(X_{l+1})$ is obtained from $\varphi(X_l)$ by summing up three numbers s_i,s_j,s_k, where i,j,k are pairwise distinct, and rearranging the sum together with s_m, $m \neq i, j, k$, in decreasing order. Write $\mathtt{s}^\circ = (\mathtt{s}_1, \ldots, \mathtt{s}_M)$ for the partition $\varphi_1(X_l)$, and let ν be uniformly distributed on $\{0, \ldots, M-1\}$, independent of X_l. By the random walk representation and Theorem 1 of [8], the law of the cyclically ordered arc lengths $(|\mathtt{s}_{1+\nu}|, \ldots, |\mathtt{s}_{M+\nu}|)$ (indices are taken modulo M) agrees with the law of the M subsequent ladder epochs of simple random walk conditioned on $H_{-M} = 2n+1$. In particular, the law of $(|\mathtt{s}_{1+\nu}|, \ldots, |\mathtt{s}_{M+\nu}|)$ is invariant under permutations and therefore equals the law of $\big(s_{\sigma(1)}, \ldots, s_{\sigma(M)}\big)$, where σ is a permutation of $\{1, \ldots, M\}$, chosen uniformly at random and independently of X_l. Note that this can also be deduced directly from the fact that X_l is uniformly distributed on the space of all l-subsets of $\{0, \ldots, 2n\}$. The probability that s_i,s_j,s_k are replaced by their sum is given by the probability that the arcs $\mathtt{s}_{\sigma^{-1}(i)+\nu}$, $\mathtt{s}_{\sigma^{-1}(j)+\nu}$, $\mathtt{s}_{\sigma^{-1}(k)+\nu}$ merge. This is the case if and only if the arcs adjoin each other and the singleton $X_{l+1} \backslash X_l$ is contained in that arc which is followed in clockwise order by the other two. More formally, the arcs merge if and only if there is a permutation ρ of the indices i, j and k such that $X_{l+1} \backslash X_l \subset \mathtt{s}_{\sigma^{-1}(\rho(i))+\nu}$, and $\sigma^{-1}(\rho(j)) = \sigma^{-1}(\rho(i)) + 1$, $\sigma^{-1}(\rho(k)) = \sigma^{-1}(\rho(i)) + 2$ (both equalities are taken modulo M). Given $X_{l+1} \backslash X_l \subset \mathtt{s}_{\sigma^{-1}(i)+\nu}$, the probability that $\mathbf{s}_{\sigma^{-1}(i)+\nu}$, $\mathbf{s}_{\sigma^{-1}(j)+\nu}$, $\mathbf{s}_{\sigma^{-1}(k)+\nu}$ merge is therefore

$$\frac{2}{M-1} \times \frac{1}{M-2}.$$

The probability that $X_{l+1} \backslash X_l \subset \mathbf{s}_{\sigma^{-1}(i)+\nu}$ is

$$\frac{\text{number of vacant sites in } \mathbf{s}_{\sigma^{-1}(i)+\nu} \text{ at time } l}{\text{total number of vacant sites at time } l} = \frac{(s_i+1)/2}{2n+1-l}.$$

Altogether, given $\varphi(X_l) = \mathbf{s}$,

$$\mathbb{P}\left(\mathbf{s}_{\sigma^{-1}(i)+\nu}, \mathbf{s}_{\sigma^{-1}(j)+\nu}, \mathbf{s}_{\sigma^{-1}(k)+\nu} \text{ merge}\right) = \left(\frac{(s_i+s_j+s_k+3)/2}{2n+1-l}\right) \frac{2}{M-1} \times \frac{1}{M-2},$$

which is the probability in (a) in the case $l < n-1$. $\square$

Theorem 2.1 now easily follows. Indeed, from the last proposition we see that $\varphi(X)$ is equal in law to the skeleton chain $\{\mathcal{X}'_k\}_{0 \le k \le n}$ started from N particles of unit mass. By the duality relation (2), reversing $\varphi(X)$ in time yields the process $\varphi(Y)$, which is the state chain of a fragmentation process.

3 Random binary forest representation

In this section, we give a second construction of the skeleton chain of the ternary coalescent in terms of random binary forests. The connection between random forests and coalescent processes was first observed by Pitman in [18]. In our description, we are guided by Chapter 5.2.3 of Bertoin [7].

3.1 Basic definitions on graphs

We first collect some basic notions on graphs which will useful for our purpose.

A (undirected) *graph* is a pair $G = (V, E)$, where V is a finite set and $E \subset \{U \subset V : |U| = 2\}$. The elements of V are called *vertices*, the elements of E *edges*. The *size* of a graph is the number of vertices $|V|$. A *subgraph* of a graph $G = (V, E)$ is a graph $H = (V', E')$ with $V' \subset V$ and $E' \subset E$.

Now let $G = (V, E)$ be a graph. Two vertices v, w are *adjacent*, if $\{v, w\} \in E$. The *degree* of a vertex v is the number of vertices adjacent to v. A sequence $(v_1, e_1, v_2, \ldots, v_m, e_m, v_{m+1})$ such that $m \ge 0$ $v_i \ne v_j$ for $i \ne j$ and $e_i = \{v_i, v_{i+1}\} \in E$ for $1 \le i \le m$ is called a *path*, or also a v_1-v_{m+1}-path. A *cycle* is a sequence $(v_1, e_1, \ldots, v_m, e_m, v_1)$ such that $m \ge 2$, $(v_1, e_1, \ldots, v_{m-1}, e_{m-1}, v_m)$ is a path and $e_m = \{v_m, v_1\} \in E$. We say that two vertices v, w are *connected*, if there exists a v-w-path. If there is a v-w-path for any $v, w \in V$, we say that the graph G is connected. The maximal connected subgraphs of G are its *connected components*. A connected graph without a cycle (as a subgraph) is called a *tree*. In a tree, a *leaf* is a vertex of degree equals 1, while the vertices of degree greater than 1 are called *internal* vertices.

We are interested in a special family of trees. A *binary tree* is either a tree consisting of a single vertex only, called the *root* of the tree, or a tree where exactly one vertex has degree 2, which we then call the root of the tree, and all the other vertices have degree 3 or they are leaves. The *height* of a vertex v in a binary tree is the number of edges of the (unique) v-r-path, where r is the root of the tree. If v is not a leaf, then there are exactly two vertices w, w' adjacent to v with height strictly bigger than that of v, the *children* of v. We call the pair $\{\{v,w\},\{v,w'\}\}$ the *outgoing edges* (from v). Finally, a *binary forest* is a graph such that its connected components are binary trees. The leaves or internal vertices of such a forest are then all those of its tree components.

Observe that a binary forest on N vertices with m tree components has $N-m$ edges, $(N+m)/2$ leaves, and $(N-m)/2$ internal vertices.

Remark 3.1. In the literature, a binary tree in our sense is often called a (rooted) *full labeled* binary tree. The term "full" reflects the fact that every vertex other than the leaves has two children, and "labeled" stresses that the vertices are distinguishable. However, we will use the term "labeled" to indicate a labeling of internal vertices.

3.2 Dynamics

Our concern here is to describe the dynamics on the space of binary forests, which will lead to another representation of the ternary coalescent.

As before let $N = 2n+1$. We consider $V = \{1, 2, \ldots, N\}$ as a set of vertices. Given a binary forest on V, we enumerate its tree components according to the increasing order of their roots.

We will assign additional labels to all internal vertices of such a forest. A *labeling* of a binary forest on V with m tree components is a bijective map from the set of $(N-m)/2$ internal vertices into $\{1, \ldots, (N-m)/2\}$. A labeled binary forest on V is then a binary forest together with a labeling. Note that internal vertices are double-labeled, by V and by the labeling just described. The set of all labeled binary forests on V with m tree components is denoted by $\mathcal{F}(m, N)$. Clearly, $\mathcal{F}(m, N)$ is empty if m is an even number.

For every $1 \le k \le n$, we define a map $R : \mathcal{F}(2k-1, N) \longrightarrow \mathcal{F}(2k+1, N)$ as follows. For each $\tau \in \mathcal{F}(2k-1, N)$, select the internal vertex with the highest label and delete both outgoing edges (and the label, since the vertex is now a leaf). We obtain a labeled binary forest with $2k+1$ trees, which we denote by $R(\tau)$.

As the reader might already guess, the map R will be the building block of the fragmentation mechanism - it breaks the tree with the highest label into three (new) trees. The reverse dynamic will correspond to the coagulation mechanism: Out of a binary forest with at least three trees, pick one leaf and connect it by adding edges to two distinct roots from other tree components. Then, three trees have merged into one (new) tree, and the selected leaf has become an internal vertex. Before underlying this procedure with randomness, let us analyze the map R in detail.

Lemma 3.1. *For every $1 \le k \le n$, the map $R : \mathcal{F}(2k-1, N) \longrightarrow \mathcal{F}(2k+1, N)$ is surjective. More precisely, for every $\tau \in \mathcal{F}(2k+1, N)$,*

$$|\{\tilde{\tau} \in \mathcal{F}(2k-1, N) : R(\tilde{\tau}) = \tau\}| = (n+k+1)k(2k-1).$$

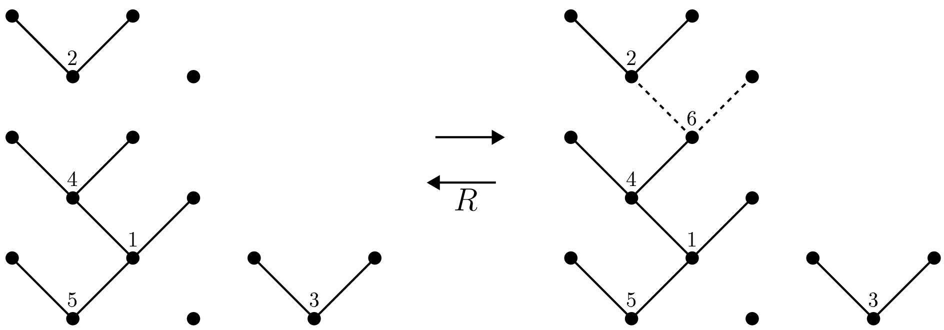

Figure 3: From left to right (right to left) one step in the coagulation (fragmentation) mechanism is shown. For simplicity, only the labeling of the internal vertices is depicted. On the left, the leaf with number 6 on the right is chosen as well as two roots from different tree components. They are connected by two edges visualized by the dashed lines on the right side.

Proof: Let $\tau \in \mathcal{F}(2k+1, N)$. In order to construct a generic $\tilde{\tau} \in R^{-1}(\tau)$, pick a leaf i from τ. Write $\rho(i)$ for the root of the tree component containing i. Then select two roots $j \neq j'$ different from $\rho(i)$, add the edges $\{i, j\}$, $\{i, j'\}$ and label the vertex i with the number $n-k+1$. Out of three components, we have obtained a new labeled binary tree with root $\rho(i)$, which is part of a forest with $2k-1$ trees. Clearly, this forest is contained in $R^{-1}(\tau)$. Moreover, different choices of i, j, j' give rise to different forests. To finish the proof, note that there are $n+k+1$ possible choices for a leaf i, and $2k(2k-1)/2$ possible choices for distinct roots $\{j, j'\}$. $\square$

Remark 3.2. Applying the map R at most n times destructs a labeled binary forest into its single vertices. Due to the recursive structure of trees, this method enables one to compute various combinatorial quantities. For example, using $|\mathcal{F}(N, N)| = 1$ and iteratively the identity

$$|\mathcal{F}(2k-1, N)| = (n+k+1)k(2k-1)\,|\mathcal{F}(2k+1, N)|$$

provided by Lemma 3.1, one obtains for $k = 2, \ldots, n+1$

$$|\mathcal{F}(2k-1, N)| = \frac{2^{k-(n+1)} n\,(2n+1)!\,(2n-1)!\,(k-2)!}{(n+k)!\,(k-1)!\,(2k-3)!}.$$

In the case $k = 1$,

$$|\mathcal{F}(1, N)| = \frac{2^{-n}(2n)!(2n+1)!}{(n+1)!} = 2^{-n}(2n+1)!\,n!\,C_n,$$

where $C_n = (2n)!/((n+1)!\,n!)$ is the nth *Catalan* number. Since there are $n!$ different labelings of internal vertices, we deduce that the number of binary trees on V is given by $2^{-n}(2n+1)!\,C_n$.

3.3 From forests to mass partitions

Denote by R^k the kth concatenation of R, where R^0 is the identity map. We randomize the input by endowing the space $\mathcal{F}(1,N)$ with the uniform probability measure and interpret the maps R^k as random variables

$$R^k : \mathcal{F}(1,N) \longrightarrow \mathcal{F}(2k+1,N), \quad k = 0,\ldots,n.$$

In words, $R^k(\tau)$ is the forest with $2k+1$ tree components which arises from $\tau \in \mathcal{F}(1,N)$ by picking the k internal vertices with the highest labels and deleting their outgoing edges. By induction, we deduce from Lemma 3.1 that R^k obeys the uniform law on the space $\mathcal{F}(2k+1)$, for each k. We then consider the random variables

$$|R^k|^{\downarrow} : \mathcal{F}(1,N) \longrightarrow \mathcal{P}^{\downarrow}_{2n+1}, \quad k = 0,\ldots,n,$$

where for a tree $\tau \in \mathcal{F}(1,N)$, $|R^k|^{\downarrow}(\tau) = \mathbf{s} = (s_1,\ldots,s_{2k+1}) \in \mathcal{P}^{\downarrow}_{2n+1}$ is the sequence of the sizes of the tree components, ranked in decreasing order.

Turning back to the ternary coalescent, let $\mathcal{X}'_k$, $k = 0,\ldots,n$, denote the skeleton chain started from N particles of unit mass. Its connection to the sizes of the tree components is given by

Proposition 3.1. *The sequence of random variables $\{|R^{n-k}|^{\downarrow}\}_{0\le k\le n}$ is the state chain of the ternary coalescent, that is*

$$\left(|R^n|^{\downarrow}, |R^{n-1}|^{\downarrow},\ldots,|R^0|^{\downarrow}\right) \stackrel{\mathrm{d}}{=} (\mathcal{X}'_0,\ldots,\mathcal{X}'_n)\,.$$

Proof: For each tree $\tau \in \mathcal{F}(1,N)$, the forest $R^n(\tau)$ has no edges, so $|R^n|^{\downarrow} = (1,\ldots,1) = \mathcal{X}'_0$. Note that given $|R^l|^{\downarrow} = \mathbf{s} = (s_1,\ldots,s_{2l+1})$ for some $1 \le l \le n$, the mass partition $|R^{l-1}|^{\downarrow}$ is obtained from $\mathbf{s}$ by replacing three elements s_i,s_j,s_k, where i,j,k are pairwise distinct, by their sum. Furthermore, observe that the random variables R^k, $l \le k \le n$, are measurable with respect to the sigma-field generated by R^l. In particular, by Proposition 2.3, the claim follows if we show that for every $0 \le l < n$, for every $\mathbf{s} = (s_1,\ldots,s_{2(n-l)+1}) \in \mathcal{P}^{\downarrow}_{2n+1}$ and for every 3-subset $\{i,j,k\} \subset \{1,\ldots,2(n-l)+1\}$,

$$\mathbb{P}\left(|R^{n-l-1}|^{\downarrow} = s^{i\oplus j\oplus k} \mid R^{n-l},\, |R^{n-l}|^{\downarrow} = \mathbf{s}\right) = \frac{s_i+s_j+s_k+3}{(2n+1-l)2(n-l)(2(n-l)-1)}.$$

Take a forest $\tau \in \mathcal{F}(2(n-l)+1,N)$. We work conditionally on $R^{n-l} = \tau$. By our observation above, R^{n-l-1} is uniformly distributed on the set of $(2n+1-l)(n-l)(2(n-l)-1)$ forests which can be obtained from τ in the way described in Lemma 3.1. We write $\tau_1,\ldots,\tau_{2(n-l)+1}$ for the tree components of τ. For every 3-subset $\{a,b,c\} \subset \{1,\ldots,2(n-l)+1\}$, the probability that the leaf i is picked in τ_a and the roots are chosen from τ_b and τ_c is therefore

$$\frac{|\tau_a|+1}{2} \times \frac{1}{(2n+1-l)(n-l)(2(n-l)-1)}.$$

Hence the probability that R^{n-l-1} evolves from τ by merging the trees τ_a, τ_b and τ_c, that is the probability that the leaf i is picked in either τ_a, τ_b or τ_c and connected to the roots of the other two components is

$$\frac{|\tau_a| + |\tau_b| + |\tau_c| + 3}{(2n+1-l)2(n-l)(2(n-l)-1)}.$$

□

As a consequence of Theorem 2.1, the time-reversed process $\{|R^k|^{\downarrow}\}_{0\le k\le n}$ is a fragmentation chain with dislocation law μ.

Remark 3.3. Adapting the proof of Corollary 5.7 in [7] to our situation, we find another way to prove Corollary 1.1, based on the binary forest representation. Namely, with $m = 2(n-l)+1$ and $\mathbf{s} = (s_1, \ldots, s_m) \in \mathcal{P}^{\downarrow}_{2n+1}$, there are

$$\frac{1}{m!}\binom{2n+1}{s_1, \ldots, s_m} = \frac{(2n+1)!}{m!\, s_1! \cdots s_m!}$$

possibilities to partition the set of vertices $\{1, \ldots, 2n+1\}$ into non-empty disjoint sets E_i, $i = 1, \ldots, m$, such that $|E_i| = s_i$ and $\min E_i < \min E_j$ for $i < j \le m$. Without labeling internal vertices, the number of binary tree structures which can be attached to E_i is $|\mathcal{F}(1, s_i)|/((s_i-1)/2)!$. Having chosen a binary tree structure for each E_i, there are $l!$ possible ways to label the l internal vertices. Recall that the tree components of a forest are enumerated in increasing order of their roots. It follows that the number of binary forests $\tau \in \mathcal{F}(m, 2n+1)$ with tree components τ_i such that $|\tau_i| = s_i$ is given by

$$\frac{(2n+1)!\, l!}{m!} \prod_{i=1}^{m} \frac{|\mathcal{F}(1, s_i)|}{s_i!\, \left(\frac{s_i-1}{2}\right)!}.$$

Since R^{n-l} is uniformly distributed on $\mathcal{F}(m, 2n+1)$, we deduce from Proposition 3.1 that

$$\mathbb{P}\left(\mathcal{X}'_l = (s_1, \ldots, s_m)\right) = \frac{\gamma(\mathbf{s})}{|\mathcal{F}(m, 2n+1)|} \frac{(2n+1)!\, l!}{m!} \prod_{i=1}^{m} \frac{|\mathcal{F}(1, s_i)|}{s_i!\, \left(\frac{s_i-1}{2}\right)!},$$

where $\gamma(\mathbf{s})$ has been defined in Section 1.2. Plugging in the values for $|\mathcal{F}(m, 2n+1)|$ and $|\mathcal{F}(1, s_i)|$ from Remark 3.2 results in the expression obtained in Corollary 1.1.

3.4 Encoding forests by paths

We conclude our discussion of binary forests by illustrating a direct connection to the random walk representation. Here, it is more convenient to consider (rooted unlabeled) plane trees and forests. In a plane forest vertices are regarded as indistinguishable, but the set of children for each vertex is ordered, as well as the set of roots of the different tree components. The ordering induces serveral natural enumerations of the vertices. For example, one of them is provided by the order in which the vertices are visited by a depth-first search, see Figure 4. More on this can be found in Chapter 6.2 of Pitman [19].

We will look at (full) binary plane forests. To relate them to the binary forests considered above, note that the number of binary plane forests on N vertices with k tree components is equal to

$$\frac{2^{(N-k)/2}k!}{N!((N-k)/2)!}|\mathcal{F}(k,N)|,$$

since there are $2^{(N-k)/2}$ possible orderings of the children of the internal vertices of a forest in $\mathcal{F}(k,N)$, $k!$ orderings of the roots, but neither vertices are labeled nor there is an additional identification of internal vertices. Clearly the ternary coalescent with a monodisperse initial configuration can also be realized on the space of binary plane forests, with the same dynamics.

There are various possibilities to code plane trees and forests by discrete functions. For a (finite) plane tree θ on N vertices, one common way is to look at its Lukasiewicz path $\{x_l\}_{0\le l\le N}$. Denoting by $v_0,\dots,v_{N-1}$ the vertices of θ listed in the order of a depth-first search and by $k(v)$ the number of children of vertex v, one defines

$$x_j = \sum_{i=0}^{j-1}(k(v_i)-1), \quad 0\le j\le N.$$

Note that $x_0=0$, $x_N=-1$, and

$$x_j - x_{j-1} = k(v_{j-1})-1, \quad 1\le j\le N. \tag{6}$$

It is easy to see that there is a bijection between Lukasiewicz paths and rooted plane trees. A sequence of such trees may then by encoded by gluing together the corresponding Lukasiewicz paths, retaining the relationship (6). In other words, the coding of the next tree starts if a new minimum is attained.

Turning to random trees, it follows from Proposition 1.4 of Le Gall [15] that a Galton-Watson tree with offspring distribution $\eta(k) = 1/2(\delta_0(k)+\delta_2(k))$, conditioned to have total progeny size N, is distributed according to a tree chosen uniformly at random among the set of all binary plane trees on N vertices. Further, the corresponding Lukasiewicz path tree is distributed as the path of simple random walk on $\mathbb{Z}$ up to time N, conditioned on $H_{-1}=N$ (see Corollary 1.6 of [15]).

We then realize that for an integer $0\le l\le n$, the path of simple random walk up to time N, conditioned on $H_{-(2l+1)}=N$, encodes a forest distributed uniformly over all binary plane forests on N vertices with $2l+1$ tree components. In particular, the sequence of the sizes of the tree components is distributed as the sequence of the ladder epochs of the conditioned random walk path, if both are put in random uniform order, say. However, the sequence of coding functions induced by the above dynamics on the space of binary forests is not directly related to the sequence of paths of the random walk representation. In this sense, the connection between the two representations is only static.

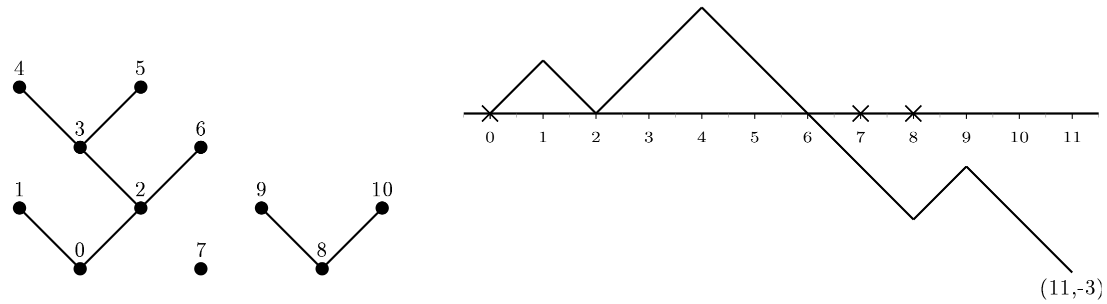

Figure 4: On the left side a binary plane forest on 11 vertices with 3 tree components is shown, where the vertices are enumerated by a depth-first search. The corresponding Lukasiewicz path is depicted on the right side. The crosses indicate where the coding of a new tree starts.

4 Asymptotics of the ternary coalescent

Having concrete realizations at hand, we are now able to investigate asymptotic properties of the ternary coalescent process. Let us write $\mathcal{X}^{[N]} = (\mathcal{X}^{[N]}(t),\, t \geq 0)$ for the coalescent with kernel κ started from the monodisperse configuration $(1, \ldots, 1)$ consisting of $N = 2n + 1$ atoms of unit mass, and put $\mathcal{X}_k'^{[N]} = \mathcal{X}^{[N]}(T_k)$, $k = 0, \ldots, n$. The number of particles at time $t \geq 0$ is denoted by $\#^{[N]}(t)$, and the number of jumps up to time t by $J^{[N]}(t)$.

We will consider the space of mass partitions with total mass bounded by 1,

$$\mathbb{S}_{\leq 1} = \left\{ \mathbf{s} = (s_1, s_2, \ldots) : s_1 \geq s_2 \geq \ldots \geq 0, \sum_{i=1}^{\infty} s_i \leq 1 \right\},$$

and the subset $\mathbb{S}_1 \subset \mathbb{S}_{\leq 1}$ of sequences with $\sum_{i=1}^{\infty} s_i = 1$. We equip $\mathbb{S}_{\leq 1}$ with the uniform distance. The induced topology coincides with that of pointwise convergence and turns $\mathbb{S}_{\leq 1}$ into a compact space. The l_1-distance induces a finer topology. However, if $(\mathbf{s}_n,\, n \in \mathbb{N})$ is some sequence in $\mathbb{S}_{\leq 1}$ converging pointwise to $\mathbf{s} \in \mathbb{S}_1$, then the convergence does also hold in the l_1-sense, as it can be easily deduced from Scheffé's lemma. Therefore, on $\mathbb{S}_1$ all these types of convergence are equivalent.

We turn to our main result of this section. Recall that the standard additive coalescent $\mathfrak{X} = (\mathfrak{X}(t),\, t \in \mathbb{R})$ is the unique additive coalescent process such that for each $t \in \mathbb{R}$, $\mathfrak{X}(t)$ has the law of the ranked sequence $\mathtt{a}_1 \geq \mathtt{a}_2 \geq \ldots$ of the atoms of a Poisson random measure on $(0, \infty)$ with intensity measure $\Lambda(da) = e^{-t} da/\sqrt{2\pi a^3}$, conditioned on $\sum_{i=1}^{\infty} \mathtt{a}_\mathtt{i} = 1$. We refer to [3] and [12] for background.

Theorem 4.1. *As $n \to \infty$, the $\mathbb{S}_1$-valued process*

$$t \mapsto \frac{1}{N} \mathcal{X}^{[N]}(e^t / N^{3/2}), \quad t \in \mathbb{R},$$

converges in the sense of finite-dimensional distributions towards the standard additive coalescent.

Here, the multiplication with $1/N$ is meant element-wise. At first glance the convergence may look surprising, since the standard additive coalescent is a *binary* coalescent that arises as a limit of additive coalescent processes as follows (Evans and Pitman [12]). Let $\mathfrak{X}^{[n]} = (\mathfrak{X}^{[n]}(t),\, t \geq 0)$ be the stochastic coalescent with binary coagulation kernel

$$\tilde{\kappa}(r,s) = r + s, \quad r, s > 0,$$

started from the monodisperse configuration with n atoms, each of mass $1/n$. Then, as $n \to \infty$, the time-shifted processes $(\mathfrak{X}^{[n]}(t + (1/2)\ln n), t \geq -(1/2)\ln n)$ converge weakly to $\mathfrak{X}$.

However, our convergence result concerns only the finite-dimensional laws. For the one-dimensional distributions, one might expect a result in this direction if one compares the one-dimensional statistics of the skeleton chains of the ternary and the additive coalescent $\mathfrak{X}^{[n]}$. The states of the additive coalescent can be expressed in terms of independent standard Borel variables (see for example (30) in [12]), which have a tail behavior similar to that of the hitting time H_k. For the finite-dimensional laws, an analysis of the first hitting time distribution shows that a "true" ternary coagulation step, i.e. the event that three particles merge which are all of a size comparable to n, only occurs with negligible probability. Therefore, under the rescaling, the process looks more like a binary coalescent.

Let us briefly comment on the scaling in the theorem. To obtain a limit for the normalized sequence of masses, the number of atoms must be of order $\sqrt{n}$. We refer to Lemma 4.2 for a better understanding. As Lemma 4.1 shows, if the process $\mathcal{X}^{[N]}$ runs for time $t/N^{3/2}$, then the amount of particles has typically reduced from N to about $\sqrt{N}/t$. Note that when approximating the standard additive coalescent with the processes $\mathfrak{X}^{[n]}$ starting from n atoms of mass $1/n$, the macroscopic picture appears at times $t + (1/2)\ln n$, at which there are about $\sqrt{n}/e^t$ particles. Here, roughly speaking, the standard Borel law plays the role of the hitting time distribution. Precise statements can be found in the books of Pitman [19], Chapter 10.3, and Bertoin [7], Chapter 5.3.

We shall present three different ways to obtain convergence for the rescaled ternary coalescent of which we discuss two in detail. The first more general method will lead to one-dimensional convergence in Proposition 4.2. It relies on the observation that the distribution of the hitting time H_k is in the domain of attraction of a stable(1/2) law. Then a size-biased reordering is used to construct the limiting mass partition. The second method resulting in finite-dimensional convergence (and therefore in the proof of the theorem) is more specialized to our situation. It is based on the identification of configurations with mass partitions via paths, as described in Section 2.1. Since the two methods do not rely on each other, the reader in a hurry may safely skip Section 4.2. In a closing remark we outline a possible third way to establish finite-dimensional convergence, using the random binary forest representation.

4.1 Number of particles

In order to relate the behavior of $\mathcal{X}^{[N]}$ to that of its skeleton chain, we prove a limit theorem for the number of particles. As just remarked, it will become clear later why we choose the spatial scale factor $N^{-1/2}$.

Lemma 4.1. *For every $t > 0$, as $n \to \infty$,*

$$\frac{\#^{[N]}(t/N^{3/2})}{\sqrt{N}} \to \frac{1}{t} \quad \textit{in probability.}$$

Proof: Using the relation $\#^{[N]}(\cdot) = N - 2J^{[N]}(\cdot)$, the claim will follow once we show that

$$\frac{J^{[N]}(t/N^{3/2})}{\sqrt{N}} - \frac{(\sqrt{N} - t^{-1})}{2} \to 0 \quad \text{in probability.} \tag{7}$$

Remember that $J^{[N]}(t/N^{3/2}) = \max\{k \in \mathbb{Z}_+ : N^{3/2} T_k \le t\}$, where T_k is the kth coagulation time given by $T_k \stackrel{\mathrm{d}}{=} \sum_{i=1}^{k} \alpha(i)^{-1} \mathbf{e}_i$, the rates $\alpha(i) = \alpha(i, N)$ are as in (1) and $\mathbf{e}_1, \mathbf{e}_2, \ldots$ is a sequence of independent standard exponential variables. Heuristically, replacing T_k by its expectation $\sum_{i=1}^{k} \alpha(i)^{-1}$, the number of jumps $J^{[N]}(t/N^{3/2})$ should roughly behave as the maximal k such that $N^{3/2} \sum_{i=1}^{k} \alpha(i)^{-1} \le t$. We will show that with the choice $k_n = n - t^{-1}\sqrt{N}/2$,

$$N^{3/2} \sum_{i=1}^{k_n} \alpha(i)^{-1} = t + o(1), \tag{8}$$

where we agree that the sum runs from 1 to the largest integer below k_n. First note that

$$\begin{aligned} N^{3/2} \sum_{i=1}^{k_n} \alpha(i)^{-1} &= N^{3/2} \sum_{i=1}^{k_n} \frac{1}{(N+1-i)(N+1-2i)(N-2i)} \\ &= N^{3/2} \left(\sum_{i=1}^{k_n} \frac{1}{(N-i)(N-2i)^2} \right) + O(n^{-1/2}). \end{aligned}$$

Furthermore, some simple computations show that for each $\varepsilon > 0$,

$$\begin{aligned} \sum_{i=1}^{k_n} \frac{1}{(N-i)(N-2i)^2} &= \int_0^{k_n} \frac{\mathrm{d}x}{(N-x)(N-2x)^2} + O(n^{-2}) \\ &= \frac{1}{N(N-2k_n)} + O(n^{-2+\varepsilon}) \\ &= \frac{t}{N^{3/2}} + O(n^{-2+\varepsilon}). \end{aligned}$$

Altogether, we obtain (8). Moreover, since

$$\operatorname{Var}\left(N^{3/2} T_{k_n}\right) = N^3 \sum_{i=1}^{k_n} \alpha(i)^{-2} = O(n^{-1/2}),$$

we deduce that $N^{3/2} T_{k_n} \to t$ in probability. From this (7) readily follows. □

4.2 Mass partitions induced by Poisson measures

We shall now prove one-dimensional convergence of the ternary coalescent process. First let us recall some basic facts about mass partitions obtained from Poisson measures, as provided in Section 2.2.3 of Bertoin [7]. Consider a measure Λ on $(0,\infty)$ such that

$$\int_0^\infty (1 \wedge x)\Lambda(dx) < \infty \quad \text{and} \quad \Lambda((0,\infty)) = \infty. \tag{9}$$

Let M be a Poisson random measure on $(0,\infty)$ with intensity Λ. From (9) it follows that M has almost surely a countably infinite number of atoms, which we may rank in decreasing order,

$$\mathtt{a}_1 \geq \mathtt{a}_2 \geq \ldots > 0.$$

Under condition (9), we further have

$$\varsigma = \sum_{i=1}^{\infty} \mathtt{a}_i < \infty \quad \text{almost surely.}$$

In our situation, Λ will be non-atomic, which implies that the atoms $\mathtt{a}_i$ are almost surely distinct. Furthermore, Λ will be of a form that guarantees the existence of a continuous density of ς,

$$\mathbb{P}(\varsigma \in dx) = \rho(x)dx, \quad x > 0,$$

with $\rho > 0$ on $(0,\infty)$.

Given some fixed $x > 0$, we want to transform the atoms $(\mathtt{a}_1, \mathtt{a}_2, \ldots)$ into a random mass-partition with total mass 1 by looking at $(\mathtt{a}_1/x, \mathtt{a}_2/x, \ldots)$ conditioned on $\sum_{i=1}^{\infty} \mathtt{a}_i = x$. In order to define the singular conditioning in a proper way, it is useful to look first at a size-biased reordering $(\mathtt{a}_i^*, i \in \mathbb{N})$ of $(\mathtt{a}_1, \mathtt{a}_2, \ldots)$. This means that conditionally on $(\mathtt{a}_1, \mathtt{a}_2, \ldots)$, we choose an index 1^* according to

$$\mathbb{P}\left(1^* = k \,|\, (\mathtt{a}_1, \mathtt{a}_2, \ldots)\right) = \mathtt{a}_k \Big/ \sum_{i=1}^{\infty} \mathtt{a}_i, \quad k \in \mathbb{N}, \tag{10}$$

set $\mathtt{a}_1^* = \mathtt{a}_{1^*}$, remove $\mathtt{a}_1^*$ from the sequence and repeat (10) with this new sequence to obtain 2^*, set $\mathtt{a}_2^* = \mathtt{a}_{2^*}$, and so on. In the following, Propositions 2.3 and 2.4 from [7] play a major role, so we summarize them for convenience in the next statement.

Proposition 4.1. (i) *Consider for each $n \in \mathbb{N} \cup \{\infty\}$ a random mass partition $S^{(n)}$ with total mass equals one almost surely, and a size-biased reordering $S^{(n)*}$ of $S^{(n)}$. Then, as $n \to \infty$, convergence in distribution of $S^{(n)}$ to $S^{(\infty)}$ in $\mathbb{S}_{\leq 1}$ is equivalent to convergence of $S^{(n)*}$ to $S^{(\infty)*}$ in the sense of finite-dimensional distributions.*

(ii) *In the setting from above, for fixed $x > 0$, the conditional law of $(\mathtt{a}_1^*, \mathtt{a}_2^*, \ldots)$ given $\varsigma \in [x, x+\varepsilon]$ has a weak limit in the sense of convergence of finite-dimensional distributions as $\varepsilon \downarrow 0$, denoted by $\mathbb{P}_x^*$, which is determined by the following Markov-type property:*

$$\mathbb{P}_x^*\left(\mathtt{a}_1^* \in dy\right) = \frac{y\rho(x-y)}{x\rho(x)}\Lambda(dy), \quad 0 < y < x,$$

and the conditional distribution of $(\mathtt{a}_2^*, \mathtt{a}_3^*, \ldots)$ *under* $\mathbb{P}_x^*$ *given* $\mathtt{a}_1^* = y$ *is* $\mathbb{P}_{x-y}^*$. *Under* $\mathbb{P}_x^*$, $\sum_{i=1}^{\infty} \mathtt{a}_i^* = x$ *almost surely.*

Having Proposition 4.1 in mind, we call a random sequence $(\mathtt{a}_1, \mathtt{a}_2, \ldots)$ that results from the decreasing rearrangement of $(\mathtt{a}_i^*, i \in \mathbb{N})$ under $\mathbb{P}_x^*$ the ranked sequence of the atoms of a Poisson random measure on $(0,\infty)$ with intensity Λ, conditioned on $\sum_{i=1}^{\infty} \mathtt{a}_i = x$. We leave it to the reader to check that $(\mathtt{a}_1^*, \mathtt{a}_2^*, \ldots)$ is then a size-biased reordering of $(\mathtt{a}_1, \mathtt{a}_2, \ldots)$, in the sense from above.

Let $\xi_1, \xi_2, \ldots$ be a sequence of independent copies of H_{-1}. Recall that by Lemma 2.1, as $l \to \infty$,

$$\mathbb{P}(\xi_1 = 2l+1) \sim \frac{1}{2}\sqrt{\frac{1}{\pi l^3}}. \tag{11}$$

For $k \in \mathbb{Z}_+$, let $\Sigma_{2k+1} = \xi_1 + \ldots + \xi_{2k+1}$, and denote by $S^{(2k+1,N)}$ a random mass partition distributed as the rearrangement in decreasing order of $\xi_1/N, \ldots, \xi_{2k+1}/N$, conditionally on $\Sigma_{2k+1} = N$. As a special case of Corollary 2.2 in [7] we have

Lemma 4.2. *Fix* $b > 0$. *Then* $S^{(2k+1,N)}$ *converges in distribution on* $\mathbb{S}_{\leq 1}$ *as* $k,\, n \to \infty$ *with* $k \sim bn^{1/2}$ *to the ranked sequence* $(\mathtt{a}_1, \mathtt{a}_2, \ldots)$ *of the atoms of a Poisson random measure on* $(0,\infty)$ *with intensity* $\Lambda(da) = b\pi^{-1/2}a^{-3/2}da$, *conditioned on* $\sum_{i=1}^{\infty} \mathtt{a}_i = 1$.

Proof: For $k \leq n$, denote by $(\xi_{1,N}^*, \ldots, \xi_{2k+1,N}^*)$ a $(2k+1)$-tuple distributed as a size-biased reordering of $(\xi_1, \ldots, \xi_{2k+1})$ given $\Sigma_{2k+1} = N$. It easily follows that for $l = 0, \ldots, n-k$,

$$\begin{aligned}
\mathbb{P}\left(\xi_{1,N}^* = 2l+1\right) &= \frac{(2k+1)(2l+1)}{N}\mathbb{P}(\xi_1 = 2l+1 \,|\, \Sigma_{2k+1} = N) \\
&= \frac{(2k+1)(2l+1)}{N}\mathbb{P}(\xi_1 = 2l+1) \times \frac{\mathbb{P}(\Sigma_{2k} = N - (2l+1))}{\mathbb{P}(\Sigma_{2k+1} = N)}.
\end{aligned}$$

If we fix $a \in (0,1)$, $b > 0$ and let l, k, n tend to infinity with $l \sim an$, $k \sim b\,n^{1/2}$, we obtain from (11)

$$\frac{(2k+1)(2l+1)}{N}\mathbb{P}(\xi_1 = 2l+1) \sim b\,\pi^{-1/2}a^{-1/2}n^{-1}. \tag{12}$$

Setting $g_k = 8\pi^{-1}k^2$, we see again by (11) that for $k \to \infty$,

$$(2k+1)\mathbb{P}(\xi_1 > g_k) \sim 1.$$

Moreover, since $k \sim b\,n^{1/2}$, we have as $k,\, n \to \infty$

$$\frac{g_k}{N} \sim \frac{4b^2}{\pi}.$$

It then follows from the theory of stable laws (see Breiman [10], Chapters 9 and 14) that

$$\frac{\Sigma_{2k+1}}{N} \to \varsigma \quad \text{in distribution as } k, n \to \infty,$$

where $\varsigma = \sum_{i=1}^{\infty} \mathtt{a}_i$ and $(\mathtt{a}_1, \mathtt{a}_2, \ldots)$ is the ranked sequence of the atoms of a Poisson random measure on $(0,\infty)$ with intensity $b\pi^{-1/2}a^{-3/2}da$. In particular, ς is stable(1/2) and has a smooth density $\rho(x) = \mathbb{P}(\varsigma \in dx)/dx$, which is strictly positive on $(0,\infty)$. Using $l \sim an$, we infer from Gnedenko's local limit theorem (see Gnedenko, Kolmogorov [13], p. 236) that

$$\frac{\mathbb{P}\left(\Sigma_{2k} = N - (2l+1)\right)}{\mathbb{P}\left(\Sigma_{2k+1} = N\right)} \sim \frac{\rho(1-a)}{\rho(1)}.$$

Together with (12) this shows

$$\mathbb{P}\left(\xi^*_{1,N} = 2l+1\right) \sim a\frac{\rho(1-a)}{n\rho(1)} b\pi^{-1/2}a^{-3/2}.$$

In particular, $\xi^*_{1,N}/N$ converges weakly as n, $k \to \infty$ with $k \sim bn^{1/2}$ towards the law

$$a\frac{\rho(1-a)}{\rho(1)} b\pi^{-1/2}a^{-3/2}da, \quad a \in (0,1).$$

Now observe that given $\xi^*_{1,N} = 2l+1$ for some $l = 0, \ldots, n-k$, we have equality in law

$$\left(\xi^*_{2,N}, \ldots, \xi^*_{2k+1,N}\right) \stackrel{\mathrm{d}}{=} \left(\xi^*_{1,N-(2l+1)}, \ldots, \xi^*_{2k,N-(2l+1)}\right).$$

By iterating the argument from above, we may therefore deduce from the second part of Proposition 4.1 that the limit law of $\left(\xi^*_{1,N}/N, \ldots, \xi^*_{2k+1,N}/N\right)$ in the sense of finite-dimensional distributions as n, $k \to \infty$ with $k \sim bn^{1/2}$ is given by the law of a size-biased reordering $(\mathtt{a}^*_1, \mathtt{a}^*_2, \ldots)$ of $(\mathtt{a}_1, \mathtt{a}_2, \ldots)$, where the latter sequence is as in the statement. Also, we have that $\sum_{i=1}^{\infty} \mathtt{a}^*_i = 1$ almost surely. From the first part of the same Proposition it then follows that the (ranked) random mass partition $S^{(2k+1,N)}$ converges in distribution to the ranked sequence of atoms $(\mathtt{a}_1, \mathtt{a}_2, \ldots)$, conditioned on $\varsigma = 1$. □

For the skeleton chain $\mathcal{X}'^{[N]}$, we derive the following consequence.

Corollary 4.1. *Fix $b > 0$. If n, $k \to \infty$ with $k \sim bn^{1/2}$, then $(1/N)\mathcal{X}'^{[N]}_{n-k}$ converges in distribution on $\mathbb{S}_1$ to the ranked sequence $(\mathtt{a}_1, \mathtt{a}_2, \ldots)$ of the atoms of a Poisson random measure on $(0,\infty)$ with intensity $\Lambda(da) = b\pi^{-1/2}a^{-3/2}da$, conditioned on $\sum_{i=1}^{\infty} \mathtt{a}_i = 1$.*

Proof: This follows from Proposition 2.1 together with the last lemma. □

Combining the corollary with the weak convergence result for the number of particles, we easily obtain one-dimensional convergence.

Proposition 4.2. *Fix $t > 0$. Then*

$$\frac{1}{N}\mathcal{X}^{[N]}(t/N^{3/2})$$

converges in distribution on $\mathbb{S}_1$ to the ranked sequence $(\mathtt{a}_1, \mathtt{a}_2, \ldots)$ of the atoms of a Poisson random measure on $(0,\infty)$ with intensity

$$\frac{t^{-1}}{\sqrt{2\pi a^3}}da, \quad a > 0,$$

conditioned on $\sum_{i=1}^{\infty} \mathtt{a}_i = 1$. *In particular, the one-dimensional distributions of the process*

$$t \mapsto \frac{1}{N}\mathcal{X}^{[N]}(e^t/N^{3/2}), \quad t \in \mathbb{R},$$

converge to those of the standard additive coalescent.

Proof: Let $k_n = n - J^{[N]}(t/N^{3/2})$. Then $\mathcal{X}^{[N]}(t/N^{3/2}) = \mathcal{X}'^{[N]}_{n-k_n}$, so we may show convergence for $(1/N)\mathcal{X}'^{[N]}_{n-k_n}$. From Lemma 4.1 it follows that as $n \to \infty$,

$$\frac{k_n}{\sqrt{n}} \to \frac{t^{-1}}{\sqrt{2}} \quad \text{in probability.} \tag{13}$$

Furthermore, we know from Corollary 4.1 that if l_n is a deterministic sequence of integers with $l_n \sim \sqrt{n}t^{-1}/\sqrt{2}$, then we have the asserted convergence for $(1/N)\mathcal{X}'^{[N]}_{n-l_n}$.

It therefore remains to argue that we may replace l_n by the random sequence k_n. To this end, recall that $\mathbb{S}_{\leq 1}$ is a compact metric space, so by Prohorov's theorem (see Billingsley [9], Section 6) the space of probability measures on $\mathbb{S}_{\leq 1}$ is relatively compact, and we only have to show convergence on $\mathbb{S}_{\leq 1}$ in the sense of finite-dimensional distributions. Since all our random mass partitions lie in $\mathbb{S}_1$ almost surely, this leads to convergence in distribution on $\mathbb{S}_1$. Denote by $x_i^{[N]}$ the ith component of $(1/N)\mathcal{X}'^{[N]}_{n-k_n}$. Finite-dimensional convergence on $\mathbb{S}_{\leq 1}$ is equivalent to say that for each $j \in \mathbb{N}$,

$$\left(x_1^{[N]}, x_1^{[N]} + x_2^{[N]}, \ldots, x_1^{[N]} + \ldots + x_j^{[N]}\right)$$

converges in distribution towards $(\mathtt{a}_1, \mathtt{a}_1 + \mathtt{a}_2, \ldots, \mathtt{a}_1 + \ldots + \mathtt{a}_j)$, where $(\mathtt{a}_1, \mathtt{a}_2, \ldots)$ is distributed as the rearrangement in decreasing order of $(\mathtt{a}_i^*, i \in \mathbb{N})$ under $\mathbb{P}_1^*$, see Proposition 4.1 (i). This follows if we show that for all $j \in \mathbb{N}$ and $\lambda_i \geq 0$, as $n \to \infty$,

$$\mathbb{E}\left[\exp\left(-\sum_{i=1}^{j} \lambda_i \left(x_1^{[N]} + \ldots + x_i^{[N]}\right)\right)\right] \to \mathbb{E}\left[\exp\left(-\sum_{i=1}^{j} \lambda_i \left(\mathtt{a}_1 + \ldots + \mathtt{a}_i\right)\right)\right]. \tag{14}$$

Denote by $f : \mathbb{S}_{\leq 1} \to (0, 1]$ the function

$$f(\mathbf{s}) = \exp\left(-\sum_{i=1}^{j} \lambda_i \left(s_1 + \ldots + s_i\right)\right), \quad \mathbf{s} = (s_1, s_2, \ldots) \in \mathbb{S}_{\leq 1}.$$

Note that $f((1/N)\mathcal{X}^{[N]}(t)) \geq f((1/N)\mathcal{X}^{[N]}(s))$ almost surely whenever $t \leq s$. By (13) we can find deterministic sequences of integers l_n^- and l_n^+ such that $l_n^- \sim l_n^+ \sim \sqrt{n}t^{-1}/\sqrt{2}$ and the probability of the event $\{l_n^- \leq k_n \leq l_n^+\}$ tends to 1 as $n \to \infty$. But on this event, we have by monotonicity

$$f\left(\frac{1}{N}\mathcal{X}'^{[N]}_{n-l_n^-}\right) \leq f\left(\frac{1}{N}\mathcal{X}'^{[N]}_{n-k_n}\right) \leq f\left(\frac{1}{N}\mathcal{X}'^{[N]}_{n-l_n^+}\right).$$

The expectations of the outer quantities converge to the right side of (4.2). This finishes the proof. □

4.3 Convergence of ladder epochs

Aldous and Pitman have shown in [3] that the exponential time change

$$F(t) = \mathfrak{X}(-\ln t), \quad t > 0,$$

with $F(0) = (1, 0, \ldots)$ transforms the standard additive coalescent into a fragmentation process which is self-similar with index $\alpha = 1/2$. In [4], Bertoin has given an explicit construction of this fragmentation process in terms of ladder epochs of Brownian excursion with drift, and our result on finite-dimensional convergence for the ternary coalescent will be based on this identity.

Let us introduce some notation. We denote by $C[0,1]$ the space of continuous real-valued paths on $[0,1]$, endowed with the uniform topology. For an arbitrary path $\omega \in C[0,1]$, its ladder time set is given by

$$\mathcal{L}(\omega) = \left\{ s \in [0,1] : \omega(s) = \inf_{[0,s]} \omega \right\}.$$

Since $\mathcal{L}(\omega)$ is a closed set, there exists a unique decomposition of $[0,1] \backslash \mathcal{L}(\omega)$ into a countable union of disjoint (open) intervals. We denote by $G(\omega)$ the ranked sequence of their lengths. By filling up with zeros, we may always interpret $G(\omega)$ as a mass partition in $\mathbb{S}_{\leq 1}$. Note that $G(\omega) \in \mathbb{S}_1$ if and only if $\mathcal{L}(\omega)$ has Lebesgue measure zero.

The construction of the dual fragmentation process F in [4] can be summarized as follows. Let $\epsilon = (\epsilon(s), 0 \leq s \leq 1)$ be a positive Brownian excursion. For every $t \geq 0$, consider the excursion dragged down with drift t, that is $\epsilon_t(s) = \epsilon(s) - st$, $0 \leq s \leq 1$, and its ladder time set $\mathcal{L}(\epsilon_t)$, which has almost surely Lebesgue measure zero. Then, the law of $(G(\epsilon_t), t \geq 0)$ and $(F(t), t \geq 0)$ coincide.

In light of our representation of the ternary coalescent in terms of ladder epochs, it seems natural to establish convergence of these objects. In this direction, the main step is to prove convergence of the underlying random paths, with the origin placed at the first instant when their minimum is attained, towards a Brownian excursion with drift.

To begin with, take a process $(J_n(t), t \geq 0)$ distributed as $(J^{[N]}(t/N^{3/2}), t \geq 0)$, and independently of this a Markov chain $\{X_l\}_{0 \leq l \leq n}$ as defined in Section 2.2. Let us first fix $t > 0$, and write $J_n = J_n(t)$. Remember that given J_n, we may identify X_{J_n} with simple random walk up to time N, conditioned to end at $-(2(n - J_n) + 1)$,

$$S(X_{J_n})_j = 2 \left(\sum_{i=0}^{j-1} X_{J_n}(i) \right) - j, \quad 0 \leq j \leq 2n+1.$$

By linear interpolation, we define the corresponding continuous random path $S_{n,t}$ on the unit interval,

$$S_{n,t}(s) = 2 \left(\sum_{i=0}^{\lfloor Ns \rfloor - 1} X_{J_n}(i) + (Ns - \lfloor Ns \rfloor) X_{J_n}(\lfloor Ns \rfloor) \right) - Ns, \quad 0 \leq s \leq 1.$$

We shall now prove convergence of the finite-dimensional laws of the $C[0,1]$-valued process $(N^{-1/2}S_{n,t},\, t>0)$. The limiting object $(B^{br}_{t^{-1}},\, t>0)$ is distributed as

$$(B^{br}_{t^{-1}},\, t>0) \stackrel{\mathrm{d}}{=} \left((B^{br}(s)-st^{-1},\, 0\le s\le 1),\, t>0\right), \tag{15}$$

where B^{br} is a standard Brownian bridge on the unit interval. In particular, for each fixed t, the distribution of $B^{br}_{t^{-1}}$ on $C[0,1]$ is that of a Brownian bridge from 0 to $-t^{-1}$.

Lemma 4.3. *The $C[0,1]$-valued process $\left(N^{-1/2}S_{n,t},\, t>0\right)$ converges in the sense of finite-dimensional distributions as $n\to\infty$ to $\left(B^{br}_{t^{-1}},\, t>0\right)$.*

Proof: Let us fix $t>0$ as above and first prove one-dimensional convergence. For $0\le s\le 1$, define

$$W_n(s) = 2\left(\sum_{i=0}^{\lfloor Ns\rfloor-1} X_n(i) + (Ns-\lfloor Ns\rfloor)X_n(\lfloor Ns\rfloor)\right) - Ns,$$

$$D_n(s) = 2\left(\sum_{i=0}^{\lfloor Ns\rfloor-1} (X_n(i)-X_{J_n}(i)) + (Ns-\lfloor Ns\rfloor)\left(X_n(\lfloor Ns\rfloor)-X_{J_n}(\lfloor Ns\rfloor)\right)\right).$$

We may then express $S_{n,t}$ as $S_{n,t}=W_n-D_n$.

The process $W_n(\cdot)$ is linear interpolation of simple random walk up to time N, conditioned to end at -1. We deduce from a conditioned version of Donsker's invariance principle (see Dwass and Karlin [11]) that $(N^{-1/2}W_n(s),\, 0\le s\le 1)$ converges weakly in $C[0,1]$ to the standard Brownian bridge B^{br}.

Concerning the drift part D_n, we let

$$\begin{aligned} D^{(1)}_n(s) &= \sum_{i=0}^{\lfloor Ns\rfloor-1} (X_n(i)-X_{J_n}(i)),\\ D^{(2)}_n(s) &= 2(Ns-\lfloor Ns\rfloor)\left(X_n(\lfloor Ns\rfloor)-X_{J_n}(\lfloor Ns\rfloor)\right), \end{aligned}$$

so that $D_n = 2D^{(1)}_n + D^{(2)}_n$. Now fix $s\in[0,1]$. A moment's thought reveals that conditioned on $J_n=n-k$ for some $k\in\{0,\dots,n\}$, the random variable $D^{(1)}_n(s)$ follows the hypergeometric distribution. More precisely,

$$\mathbb{P}\left(D^{(1)}_n(s)=j \mid J_n=n-k\right) = \frac{\binom{\lfloor Ns\rfloor}{j}\binom{N-\lfloor Ns\rfloor}{k-j}}{\binom{N}{k}},$$

where $\max\{0,k+\lfloor Ns\rfloor-N\}\le j\le\min\{k,\lfloor Ns\rfloor\}$. As a consequence,

$$\mathbb{E}\left[D^{(1)}_n(s)\mid J_n=n-k\right] = k\frac{\lfloor Ns\rfloor}{N},\quad \mathrm{Var}\left(D^{(1)}_n(s)\mid J_n=n-k\right)\le k. \tag{16}$$

Let $k_n = n - J_n$. Choosing $\varepsilon > 0$ arbitrarily small, we have for large n by the law of total probability

$$\begin{aligned} &\mathbb{P}\left(N^{-1/2}|D_n(s) - 2k_n s| > \varepsilon\right) \\ \le & \sum_{k=0}^{\lfloor \sqrt{n}t^{-1}\rfloor} \mathbb{P}\left(N^{-1/2}|D_n^{(1)}(s) - \mathbb{E}[D_n^{(1)}(s)| > \varepsilon/3 \mid k_n = k\right) \mathbb{P}(k_n = k) \\ &+ \mathbb{P}\left(k_n \ge \sqrt{n}t^{-1}\right) \\ = & \; o(1), \end{aligned}$$

where the last line follows from (13), (16) and Chebyshev's inequality. Since by (7), $N^{-1/2}2k_n s$ converges in probability to $t^{-1}s$, so does $N^{-1/2}D_n(s)$. In particular, the finite-dimensional laws of $(N^{-1/2}D_n(s),\, 0 \le s \le 1)$ converge to those of $(t^{-1}s,\, 0 \le s \le 1)$. Moreover, $D_n(s)$ is increasing in s, and a similar computation entails that for λ large enough, as $n \to \infty$,

$$\mathbb{P}\left(N^{-1/2}D_n(1) \ge \lambda\right) = o(1).$$

By Theorem 8.4 of Billingsley [9], we conclude that the distributions of $N^{-1/2}D_n(\cdot)$ form a tight sequence. It follows that $(N^{-1/2}D_n(s),\, 0 \le s \le 1)$ converges in probability to $(t^{-1}s,\, 0 \le s \le 1)$. Applying now Theorem 4.4 from [9] together with the continuous mapping theorem finishes the proof of the one-dimensional convergence.

The arguments obviously extend to finite-dimensional distributions. Indeed, the bridge term W_n is the same for all t, and the drift term D_n converges in probability, for each t. Therefore, finite-dimensional convergence follows again from Theorem 4.4 of [9]. $\square$

As for discrete paths, we introduce for $v \in [0,1]$ the shift operator θ on $C[0,1]$,

$$(\theta_v\omega)(s) = \begin{cases} \omega(s+v) - \omega(v) & ,\ 0 \le s \le 1-v \\ \omega(s+v-1) - \omega(v) + \omega(1) - \omega(0) & ,\ 1-v < s \le 1 \end{cases}.$$

Define $H : C[0,1] \to [0,1]$ as the first time when the global minimum is attained,

$$H(\omega) = \inf\left\{ s \in [0,1] : \omega(s) = \inf_{[0,1]} \omega \right\}.$$

Clearly, H is not continuous on the whole space, but it is so restricted to the subset of paths which uniquely attain their minimum. It is well-known and also implied by the subsequent Lemma 4.4 that the distribution of $B^{br}_{t^{-1}}$ is fully supported on this subset. Further, the shift operator is continuous as a map $\theta : C[0,1] \times [0,1] \to C[0,1]$, $\theta(\omega, v) = \theta_v\omega$. Setting $\theta_H\omega = \theta_{H(\omega)}\omega$, it then follows from the above lemma and the continuous mapping theorem that for $n \to \infty$,

$$\left(N^{-1/2}\theta_H S_{n,t},\, t > 0\right) \to \left(\theta_H B^{br}_{t^{-1}},\, t > 0\right)$$

in the sense of finite-dimensional distributions. Recall $(B^{br}_{t^{-1}}(s),\, 0 \le s \le 1) \stackrel{\mathrm{d}}{=} (B^{br}(s) - st^{-1},\, 0 \le s \le 1)$, where B^{br} is a Brownian bridge (the same for all t). Denoting by ϵ a standard Brownian excursion, it has been proven by Vervaat in [20] that

$$\theta_H B^{br} \stackrel{\mathrm{d}}{=} \epsilon.$$

Since $\theta_u \circ \theta_v = \theta_w$ for $w = u + v[\mathrm{mod}\ 1]$, we have $\theta_H = \theta_H \circ \theta_v$ pathwise for every $0 \le v \le 1$. Therefore, if μ denotes the almost surely unique instant when B^{br} attains its minimum,

$$\begin{aligned} \theta_H B^{br}_{t^{-1}} &\stackrel{\mathrm{d}}{=} \theta_H \circ \theta_\mu \left(B^{br} - st^{-1},\, 0 \le s \le 1\right) \\ &= \theta_H \left(\theta_H B^{br} - st^{-1},\, 0 \le s \le 1\right) \\ &\stackrel{\mathrm{d}}{=} \theta_H \epsilon_{t^{-1}}. \end{aligned} \tag{17}$$

Here, as above, $\epsilon_{t^{-1}}(s) = \epsilon(s) - st^{-1}$ is the Brownian excursion dragged down with drift t^{-1}. Since $\epsilon_{t^{-1}}$ attains its minimal value almost surely at the endpoint, we have proven the following

Corollary 4.2. *In the notation above, $\left(N^{-1/2}\theta_H S_{n,t},\, t > 0\right)$ converges in the sense of finite-dimensional distributions as $n \to \infty$ to $(\epsilon_{t^{-1}},\, t > 0)$.*

The convergence of the ternary coalescent is now easy to establish. As last preparation, let us recall a technical result. Call a point $x \in [0,1]$ a *local minimum* of $\omega \in C[0,1]$, if there exists $\delta > 0$ such that for all $y \in [\max\{x-\delta, 0\}, \min\{x+\delta, 1\}]$, $\omega(x) \le \omega(y)$. The following statement is true for all real t.

Lemma 4.4. *With probability one, all local minima of $(\varepsilon_t(s),\, 0 \le s \le 1)$ are distinct.*

Proof: By (17), we may show the statement for $(B^{br}_t(s),\, 0 \le s \le 1)$ instead. Since for the time-reversed process, it holds that

$$(B^{br}(1-s) - (1-s)t,\, 0 \le s \le 1) \stackrel{\mathrm{d}}{=} (B^{br}(s) + st - t,\, 0 \le s \le 1),$$

it suffices to show that for some $1/2 \le r < 1$, $(B^{br}_t(s),\, 0 \le s \le r)$ has almost surely distinct local minima. However, if $\mathcal{F}_r$ denotes the filtration generated by the canonical process $x.$ on $C[0,1]$ up to time $r < 1$, $\mathbb{Q}$ denotes the law of B^{br}_t and, for a moment, $\mathbb{P}$ is Wiener measure and p the Gaussian transition kernel, it is well-known that $\mathbb{Q}$ is locally absolute continuous with respect to $\mathbb{P}$,

$$\mathbb{Q}|_{\mathcal{F}_r} = \frac{p_{1-r}(x_r, -t)}{p_1(0,-t)} \cdot \mathbb{P}|_{\mathcal{F}_r}.$$

Since the local minima of Brownian motion on $[0,1]$ are distinct almost surely (see for example Theorem 2.11 in the book of Mörters and Peres [16]), the lemma is proven. □

Proof of Theorem 4.1: In view of Bertoin's result in [4], the claim follows if we show that the finite-dimensional laws of

$$t \mapsto \frac{1}{N}\mathcal{X}^{[N]}(t/N^{3/2}), \quad t > 0,$$

converge to those of $(G(\varepsilon_{t^{-1}}),\, t > 0)$. Remember the map φ constructed in Section 2.1 sending configurations to mass partitions. With $J_n(t) = J^{[N]}(t/N^{3/2})$ defined as above, we have already seen that

$$\left(\frac{1}{N}\varphi(X_{J_n(t)}),\, t \geq 0\right) \stackrel{\mathrm{d}}{=} \left(\frac{1}{N}\mathcal{X}^{[N]}(t/N^{3/2}),\, t \geq 0\right).$$

Let $t > 0$, and assume that conditionally on J_n,

$$\frac{1}{N}\varphi(X_{J_n(t)}) = (s_1, \ldots, s_{2(n-J_n(t))+1}),$$

where $Ns_i \in \{1, 3, 5, \ldots, N\}$ with $\sum s_i = 1$. Then by construction of both φ, G and linear interpolation,

$$G(N^{-1/2}\theta_H S_{n,t}) = (g_1, \ldots, g_{2(n-J_n(t))+1}),$$

with $g_i = s_i - 1/N$ for all i. Thus, the theorem follows if we show finite-dimensional convergence of $(G(N^{-1/2}\theta_H S_{n,t}),\, t > 0)$ to $(G(\varepsilon_{t^{-1}}),\, t > 0)$. It is easy to check that $G : C[0,1] \to \mathbb{S}_{\leq 1}$ is continuous on the subset of those paths which attain their local minima at unique points. By Lemma 4.4, the distribution of $\varepsilon_{t^{-1}}$ assigns mass one to this subset. Therefore, Corollary 4.2 and the continuous mapping theorem yield convergence of the finite-dimensional distributions on $\mathbb{S}_{\leq 1}$, and since $G(\varepsilon_{t^{-1}}) \in \mathbb{S}_1$ with probability one, we obtain finite-dimensional convergence on $\mathbb{S}_1$. $\square$

Concluding remarks

(i) For the proof of Theorem 4.1 we used the random walk representation. Let us point out another possibility to derive convergence, using the random binary forest representation. Following the construction in Section 3, the state chain of the ternary coalescent starting from N particles of unit mass can be realized in reversed time by deleting successively pairs of outgoing edges from a random tree uniformly distributed over all binary plane trees on N vertices. Such a random tree can be seen as a Galton-Watson tree with offspring distribution $\mu(k) = \frac{1}{2}(\delta_0(k)+\delta_2(k))$, conditioned to have total population size N. One finds oneself in the setting of Theorem 23 (in the sublattice case) of Aldous [2]. In particular, if $\tau^{[N]}$ denotes the uniform binary plane tree on N vertices, where mass $1/N$ is assigned to each vertex and the edges are rescaled to have length $1/\sqrt{N}$, then $\tau^{[N]}$ converges weakly as $N \to \infty$ to the Brownian continuum random tree (CRT) introduced in [1]. By splitting the skeleton of this tree into subtrees according to a Poisson process of cuts with some intensity $t \geq 0$ per unit length, Aldous and Pitman [3] derived from the CRT an $\mathbb{S}_1$-valued fragmentation process of ranked masses

of tree components, indexed by the intensity t. Further, they showed that the time change $t \mapsto e^{-t}$ turns this process into the standard additive coalescent. Similar to [3], it should be possible to approximate the Poisson process of marks on the CRT by the process of deleting edges from the binary plane tree. This would lead to another proof of Theorem 4.1.
(ii) Recall the random walk representation introduced in Section 2. Fix an integer k of size at least 3, and define the configuration space $\mathcal{C}_n^k$ as the set of all subsets of $\{0, \ldots, (k-1)n\}$ with cardinality less or equal to n. Now identify a configuration $x \in \mathcal{C}_n^k$ with a path of a walk that goes up $k-2$ steps if a site is occupied and one step down otherwise, i.e. $S^{(k)}(x)_0 = 0$ and for $1 \leq j \leq (k-1)n+1$,

$$S^{(k)}(x)_j = k\left(\sum_{i=0}^{j-1} x(i)\right) - j.$$

By imposing an analogous dynamics, i.e. by occupying successively n sites chosen uniformly at random from $\{0, \ldots, (k-1)n\}$, the sequence of ladder epochs of the corresponding new paths is now a realization of the state chain of the k-ary coalescent process with kernel $\kappa_k(r_1, \ldots, r_k) = r_1 + \ldots + r_k + k/(k-2)$, started from $(k-1)n+1$ particles of unit mass. As for the case $k = 3$, running this process backwards in time yields a fragmentation process. Moreover, Kemperman's formula applies also to first hitting times of such asymmetric random walks, so that their distributions can easily be computed. With some minor modifications, and under a different rescaling of time, one again obtains convergence of the finite-dimensional laws of this k-ary coalescent process towards those of the standard additive coalescent.

Not surprisingly, there is an analogous random $(k-1)$-ary forest representation of this process. Indeed, when glueing (full) $(k-1)$-ary trees by picking uniformly at random one leaf and $k-1$ roots from different components, in a similar way to Section 3 for the case $k = 3$, the ranked sequence of the tree sizes is another realization of the state chain of the k-ary coalescent with kernel κ_k.

This remark shows that our ternary coalescent process is only one particular process out of a family of k-ary coalescents that can be studied by the same means.

Acknowledgments

I am grateful to Jean Bertoin for introducing me to the topic and for helpful advice. Further I would like to thank two anonymous referees for their valuable comments.

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Hydrodynamic limit of a ternary coalescent process and solution to Smoluchowski's coagulation equations

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Abstract

We study the hydrodynamic behavior of a particular ternary stochastic coalescent which admits a representation in terms of hitting times of simple random walk. For monodisperse initial configurations we obtain explicit limits of the particle concentrations and show that they provide a solution to Smoluchowski's coagulation equations.

Subject classifications: 60K35; 82C23.
Key words: Smoluchowski coagulation equations, hydrodynamic limit, ternary coalescent, hitting times.

0 Introduction

The purpose of this note is to discuss the hydrodynamic limit for the ternary coalescent process introduced in [3], that is the stochastic coalescent with ternary coagulation kernel

$$\kappa(k, l, m) = k + l + m + 3, \quad k, l, m > 0.$$

This process describes the merging of particles in a medium as time passes. More specifically, consider an odd number $N = 2n + 1$ of particles (or atoms) with strictly positive masses. Then, three particles of masses k, l, m, say, coalesce into a single particle of mass $k + l + m$ at rate $\kappa(k, l, m)$. After n steps, only one big particle is left over, and the system remains in this state.

We henceforth assume that at time zero, the system consists out of N masses of size one. The hydrodynamic behavior concerns the evolution in time of the concentration of particles of mass k when N tends to infinity. In [3], the ternary coalescent was constructed by means of ladder epochs of a conditioned simple random walk. This representation allows us to compute limits for the concentrations which solve Smoluchowski's coagulation equations for the ternary coagulation kernel κ.

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In 1916, Smoluchowski [11] introduced a mean-field model to study the evolution in time of the concentration of particles $c_t(k)$ carrying a certain (discrete) mass $k = 1, 2, \ldots$, where two particles of masses l, m, say, merge into a single atom of mass $l + m$ when colliding. He derived the system of equations ((67) on page 584 in [11])

$$\frac{\mathrm{d}}{\mathrm{d}t} c_t(k) = \frac{1}{2} \sum_{l<k} c_t(l) c_t(k-l) \tilde{\kappa}(l, k-l) - c_t(k) \sum_{l=1}^{\infty} c_t(l) \tilde{\kappa}(k, l), \tag{1}$$

where $\tilde{\kappa}$ is a positive symmetric rate kernel determined by physical quantities like radius of operation or diffusivity constant.

In our model, three particles are involved in a coagulation event. Taking into account a monodisperse initial configuration, the natural generalization of the equations for a nonnegative symmetric ternary kernel as in our case reads ($k \in \mathbb{N}$)

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} c_t(k) & = & \dfrac{1}{6} \displaystyle\sum_{\substack{l,m \in \mathbb{N} \\ l+m<k}} c_t(l) c_t(m) c_t(k-l-m) \kappa(l, m, k-l-m) \\ & & - \dfrac{1}{2} c_t(k) \displaystyle\sum_{l,m=1}^{\infty} c_t(l) c_t(m) \kappa(k, l, m), \\ c_0(k) & = & \mathbb{1}_{\{k=1\}}. \end{cases} \tag{2}$$

Here, the term with prefactor 1/6 on the right side reflects the creation of particles of mass k out of three particles of smaller sizes, an effect that increases the concentration, whereas the term with prefactor 1/2 stands for the depletion of particles of mass k after coagulation with two other particles. The factors 1/6 and 1/2 are due to the symmetry of the kernel κ.

Both (1) and (2) fall into the category of infinite-dimensional nonlinear evolution equations. In general, such systems are not exactly solvable, and existence or not-existence of (local) solutions depend on the initial data as well as on the form of the coagulation kernel. In case of system (1), explicit solutions can for example be computed for the kernels $\tilde{\kappa}(i, j) = 1$, $\tilde{\kappa}(i, j) = i+j$ and $\tilde{\kappa}(i, j) = ij$, and also for linear combinations of these three types, see Spouge in [12]. In the monodisperse setting, a probabilistic interpretation of the solutions for these three fundamental models was given in terms of branching processes, see e.g. Deaconu and Tanré in [6]. The overview of Aldous [1] includes further references.

Our approach to approximate the equations (2) by a finite particle system has already proven successful for a wide class of binary coagulation kernels, the model there being the Marcus-Lushnikov coalescent. Norris [10] gives a general statistical derivation of the equations. Extensions to k-nary coagulation kernels can be found in Kolokoltsov [9]. A detailed probabilistic treatment leading to explicit solutions of the coagulation equations for (sub)multiplicative and additive binary kernels is given in the book of Bertoin [4]. This source also served as the guiding line for our work.

The rest of this note is organized as follows. In the next part we define the ternary coalescent and state the main result. Afterward we collect some statements about the coalescent that appeared in [3]. Then we prove a limit law for the total concentration of particles as well as a joint limit law for the sizes of a pair of atoms. This enables us to prove the main result. We finish with some general remarks.

1 The hydrodynamic behavior

Throughout this text, let $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$.

1.1 Definition of the stochastic coalescent and main result

The state space of the coalescent process will be a subset of the space

$$\mathcal{S}^{\downarrow} = \{\mathbf{s} = (s_1, s_2, \ldots) : s_1 \geq s_2 \geq \ldots \geq 0, s_k = 0 \text{ for } k \text{ sufficiently large}\},$$

endowed with the uniform distance. The components of a sequence $\mathbf{s} \in \mathcal{S}^{\downarrow}$ are commonly interpreted as particle (or atom) sizes. Zeros represent no particles, so we usually drop them from notation and write $\mathbf{s} = (s_1, \ldots, s_m)$ if the strictly positive elements of $\mathbf{s} \in \mathcal{S}^{\downarrow}$ are given by $s_1, \ldots, s_m$. If $\mathbf{s} = (s_1, s_2, \ldots) \in \mathcal{S}^{\downarrow}$ and $1 \leq i < j < k$, $\mathbf{s}^{i \oplus j \oplus k}$ denotes the sequence in $\mathcal{S}^{\downarrow}$ obtained from $\mathbf{s}$ by merging its ith, jth and kth terms, that is one removes s_i, s_j, s_k and rearranges the remaining elements together with the sum $s_i + s_j + s_k$ in decreasing order. The following definition is from [3].

Definition 1.1. The ternary coalescent with values in $\mathcal{S}^{\downarrow}$ and kernel κ is a continuous time Markov process $\mathcal{X} = (\mathcal{X}(t), t \geq 0)$ with state space $\mathcal{S}_{\prime}^{\downarrow}$ for an appropriate subset $\mathcal{S}_{\prime}^{\downarrow}$ of $\mathcal{S}^{\downarrow}$, and jump rates

$$q(\mathbf{s}, \cdot) = \sum_{1 \leq i < j < k,\, s_k > 0} \kappa(s_i, s_j, s_k) \delta_{\mathbf{s}^{i \oplus j \oplus k}}.$$

We use the notation $\mathcal{X}^{[N]} = (\mathcal{X}^{[N]}(t),\, t \geq 0)$ to indicate the coalescent started from $N = 2n + 1$ atoms $(1, \ldots, 1)$ of mass one. For $t \geq 0$ and $k \in \mathbb{N}$, put

$$c_t^{[N]}(k) = \frac{1}{N} \left| \left\{ i \in \mathbb{N} : \mathcal{X}_i^{[N]}(t/N^2) = k \right\} \right|,$$

where $\mathcal{X}_i^{[N]}$ denotes the ith component of $\mathcal{X}^{[N]}$. Let us briefly comment on the scaling. Since we look for particle concentrations per unit volume, we scale space by the factor $1/N$. Concerning time, note that at all times t there should be on the order of N particles in the system. Therefore, in any finite time interval the number of mergings of a tagged particle should be of constant order. Since there are on the order of N^2 possibilities for a particle to coalesce, this leads to rescale the coagulation kernel by $1/N^2$, that is to look at time t/N^2.

In the limit $N \to \infty$, $c_t^{[N]}(k)$ will converge towards a deterministic quantity $c_t(k)$ which we introduce next. In this direction, let $0 < p < 1$, $q = 1 - p$, and define a generalized binomial probability law ν_p supported on the odd positive integers,

$$\nu_p(k) = \frac{1}{k} \binom{k}{(k+1)/2} p^{(k+1)/2} q^{(k-1)/2}, \quad k \in \mathbb{N} \text{ odd.} \tag{3}$$

Note that ν_p is the law of the first hitting time

$$H_{-1}(\zeta) = \inf\{m \geq 1 : \zeta_m = -1\}$$

of a random walk $\zeta = \{\zeta_m\}_{m\geq 0}$ on $\mathbb{Z}$ with $\zeta_0 = 0$ and independent increments satisfying $\mathbb{P}(\zeta_i - \zeta_{i-1} = 1) = q$, $\mathbb{P}(\zeta_i - \zeta_{i-1} = -1) = p$, for $i \in \mathbb{N}$. One can check that ν_p admits finite moments if and only if $p > 1/2$. In this case, its mean is given by $(2p-1)^{-1}$.

The reason why the following two functions play an important role will be clear from Lemma 1.2. For $t \geq 0$, set

$$\psi(t) = \ln\left(\frac{2}{2+t}\right) + t.$$

Note that $\psi(t)$ is smooth and strictly increasing in t, with $\psi(0) = 0$. We denote its inverse by $\varphi(s) = \psi^{-1}(s)$, $s \geq 0$. Finally, let

$$p(t) = \frac{2+\varphi(t)}{2(1+\varphi(t))} \tag{4}$$

and define

$$c_t(k) = \frac{\nu_{p(t)}(k)}{1+\varphi(t)}.$$

Note that for even k, $c_t(k) = 0$ for all t, while for odd k,

$$c_t(k) = \frac{\binom{k}{(k+1)/2}}{(1+\varphi(t))k}\left(\frac{2+\varphi(t)}{2(1+\varphi(t))}\right)^{(k+1)/2}\left(\frac{\varphi(t)}{2(1+\varphi(t))}\right)^{(k-1)/2}.$$

Remark 1.1. The initial condition already tells us that necessarily $c_t(k) = 0$ for all even k and all $t \geq 0$. In terms of the stochastic coalescent, this is reflected in the fact that no particles of even size appear.

Our goal is to prove the following

Theorem 1.1. *Let $t \geq 0$, and $k \in \mathbb{N}$. For every $p \in \mathbb{N}$, as $n \to \infty$,*

$$c_t^{[N]}(k) \to c_t(k) \quad \text{in } L^p.$$

Furthermore, $(c_t(k), k \in \mathbb{N}, t \geq 0)$ solves Smoluchowski's coagulation equations (2) for the kernel κ with initial condition $c_0(k) = \mathbb{1}_{\{k=1\}}$.

1.2 Properties of the ternary coalescent

We recall some facts about $\mathcal{X}^{[N]}$ from [3]. To this end, let us introduce some additional notation. For every $k = 0, \ldots, n+1$, let T_k be the instant of the kth coagulation, with the convention $T_0 = 0$, $T_{n+1} = \infty$. The number of particles present at time $t \geq 0$ is denoted by $\#^{[N]}(t)$, and number of jumps up to time t by $J^{[N]}(t)$. Note that when time passes, $\#^{[N]}(t)$ decreases by steps of 2 from N to 1, and the coalescent attains n different states, which are given by $\mathcal{X}^{[N]}(T_k)$, $k = 0, \ldots, n$.

Lemma 1.1. *In the preceding notation, the following holds true.*

(i) *The sequence $\Delta_k = T_k - T_{k-1}$, $k = 1, \ldots, n$, of the waiting times between two coagulations is a sequence of independent exponential variables with respective parameters*

$$\alpha(k) = (N + 1 - k)(N + 1 - 2k)(N - 2k).$$

In particular, the sequences $\{T_k\}_{0 \le k \le n}$ and $\{\mathcal{X}^{[N]}(T_k)\}_{0 \le k \le n}$ are independent.

(ii) *The sequence $\{\mathcal{X}^{[N]}(T_k)\}_{0 \le k \le n}$ is a Markov chain with one-dimensional distributions given by*

$$\mathbb{P}\left(\mathcal{X}^{[N]}(T_l) = \mathbf{s}\right) = \mathbb{P}\left((\xi_{(N-2l)}, \ldots, \xi_{(1)}) = (s_1, \ldots, s_{N-2l}) \,|\, \xi_1 + \ldots + \xi_{N-2l} = N\right),$$

where $0 \le l \le n$, $\mathbf{s} = (s_1, \ldots, s_{N-2l}) \in \mathcal{S}^{\downarrow}$, the ξ_i are $N - 2l$ independent copies of the hitting time $H_{-1} = \inf\{m \ge 1 : \zeta_m = -1\}$ of simple random walk $\zeta = \{\zeta_m\}_{m \ge 0}$ on $\mathbb{Z}$ starting from the origin, and $\xi_{(k)}$ denotes the kth order statistic of $\xi_1, \ldots, \xi_{N-2l}$.

Proof: The first statement is Proposition 1.1 (i) in [3], and the second follows from the remark below Corollary 1.1 in the same paper. □

Note that the random variables ξ_i in part (ii) of the lemma are distributed according to $\nu_{1/2}$. However, one could take any law ν_p since the conditional distribution of the ξ_i given their total sum does not depend on the parameter $0 < p < 1$.

1.3 The total concentration of particles per unit volume

Similarly to Lemma 5.1 in [3] we prove a weak limit law for the number of particles when N tends to infinity, but here with spatial scaling $1/N$ and time scaling $1/N^2$, as it was already motivated above. Recall the definitions of ψ and its inverse φ from the beginning.

Lemma 1.2. *For every $t \ge 0$, as $n \to \infty$,*

$$\sum_{k=1}^{\infty} c_t^{[N]}(k) = \frac{\#^{[N]}(t/N^2)}{N} \to \frac{1}{1 + \varphi(t)} \quad \textit{in probability.}$$

Proof: Since $\#^{[N]}(\cdot) = N - 2J^{[N]}(\cdot)$, the statement is equivalent to

$$\frac{J^{[N]}(\psi(t)/N^2)}{N} - \left(\frac{t}{2(t+1)}\right) \to 0 \quad \text{in probability as } n \to \infty,$$

for each $t \ge 0$. In order to prove that, we make use of the identity

$$J^{[N]}(\psi(t)/N^2) = \max\{k \ge 0 : N^2 T_k \le \psi(t)\}.$$

Note that $T_k \stackrel{\mathrm{d}}{=} \sum_{i=1}^{k} \alpha(i)^{-1}\mathbf{e}_i$, where $\mathbf{e}_1, \mathbf{e}_2, \ldots$ is a sequence of independent standard exponential variables and the rates $\alpha(i) = \alpha(i, N)$ are from Lemma 1.1 (i). We fix $t \geq 0$ and put

$$k_n = \left(\frac{t}{t+1}\right) n.$$

We claim that

$$N^2 \sum_{i=1}^{k_n} \alpha(i)^{-1} = \psi(t) + O(n^{-1}), \tag{5}$$

where the sum is meant to run from 1 to the largest integer below k_n. First,

$$\begin{aligned} N^2 \sum_{i=1}^{k_n} \alpha(i)^{-1} &= N^2 \sum_{i=1}^{k_n} \frac{1}{(N+1-i)(N+1-2i)(N-2i)} \\ &= N^2 \left(\sum_{i=1}^{k_n} \frac{1}{(N-i)(N-2i)^2} \right) + O(n^{-1}). \end{aligned}$$

We may replace the sum by

$$\sum_{i=1}^{k_n} \frac{1}{(N-i)(N-2i)^2} = -\frac{1}{4} \int_0^{k_n} \frac{\mathrm{d}x}{(x-N)(x-N/2)^2} + O(n^{-3}).$$

A computation of the integral gives

$$\begin{aligned} & -\frac{1}{4} \int_0^{k_n} \frac{\mathrm{d}x}{(x-N)(x-N/2)^2} \\ = \; & \frac{1}{N^2} \left(\ln\left(\frac{N-2k_n}{N-k}\right) + \ln\left(\frac{N-1}{N-2}\right) \right) + \frac{1}{N} \left(\frac{1}{N-2k} - \frac{1}{N-2} \right). \end{aligned}$$

Plugging in the definition of k_n, this implies

$$N^2 \sum_{i=1}^{k_n} \alpha(i)^{-1} = t + \ln\left(\frac{2}{2+t}\right) + O(n^{-1}),$$

which is (5) by definition of ψ. Bounding the sum over $\alpha(i)^{-2}$ by a constant times N^{-5}, we obtain for the variance

$$\mathrm{Var}\left(N^2 T_{k_n}\right) = N^4 \sum_{i=1}^{k_n} \alpha(i)^{-2} = O(n^{-1}).$$

This proves that $N^2 T_{k_n} \to \psi(t)$ in probability. Now, the claim of the lemma easily follows. $\square$

For the statement of the next lemma, we denote by $(\theta_1^{[N]}(t), \theta_2^{[N]}(t))$ a pair of atoms chosen uniformly at random and without replacement from the non-zero components of

$\mathcal{X}^{[N]}(t)$. More precisely, if $t \geq 0$, the conditional law of $(\theta_1^{[N]}(t), \theta_2^{[N]}(t))$ given $\mathcal{X}^{[N]}(t) = (s_1, \ldots, s_m)$ is that of $(s_{\sigma(1)}, s_{\sigma(2)})$, where σ is a permutation of the numbers $\{1, \ldots, m\}$ chosen uniformly at random and independently of the state $\mathcal{X}^{[N]}(T_{(N-m)/2})$. Using the connection of the ternary coalescent to hitting times, we obtain the following joint limit theorem.

Lemma 1.3. *Let $t \geq 0$. As $n \to \infty$,*

$$\left(\theta_1^{[N]}(t/N^2), \theta_2^{[N]}(t/N^2)\right) \to (\theta_1, \theta_2) \quad \textit{in distribution,}$$

where (θ_1, θ_2) is a pair of independent random variables, each with law $\nu_{p(t)}$.

Proof: Let $(\xi_i,\, i \in \mathbb{N})$ be a sequence of independent random variables with law $\nu_{1/2}$. By Lemma 1.1 (ii), conditionally on $\#^{[N]}(t/N^2) = k \geq 3$, the pair $(\theta_1^{[N]}(t/N^2), \theta_2^{[N]}(t/N^2))$ has the same distribution as (ξ_1, ξ_2) given $\xi_1 + \ldots + \xi_k = N$. From (3) we know how this distribution looks like, and it is easy to see that given

$$\#^{[N]}(t/N^2) = \left(\frac{1}{1+\varphi(t)} + o(1)\right) N \quad \text{with } o(1) \to 0 \text{ as } N \to \infty,$$

the pair $(\theta_1^{[N]}(t/N^2), \theta_2^{[N]}(t/N^2))$ converges in law towards a pair (θ_1, θ_2) of independent random variables with law $\nu_{p(t)}$. In combination with Lemma 1.2, the claim follows. □

1.4 Solution to Smoluchowski's coagulation equations

This part is devoted to the proof of Theorem 1.1. Let us first give the intuitive reasoning behind the convergence. From Lemma 1.2 we know that the total particle concentration at time t/N^2 behaves as $(1+\varphi(t))^{-1}$. Put $M = (1+\varphi(t))^{-1}N$ and assume M is an odd positive number. By Lemma 1.1 (ii), $\mathcal{X}^{[N]}(t/N^2)$ conditioned to have exactly M non-zero elements is distributed according to the ranked sequence of M independent copies ξ_i of the hitting time H_{-1} of simple random walk, conditionally on $\xi_1 + \ldots + \xi_M = N$. Given their total sum equals N, each ξ_i should be on average of size $N/M = 1 + \varphi(t)$. By Lemma 1.3, as n tends to infinity, the influence of the conditioning vanishes if the law of the ξ_i is changed to $\nu_{p(t)}$, where $p(t)$ is defined in (4). Note that the mean of $\nu_{p(t)}$ is precisely $1 + \varphi(t)$. The concentration of particles with mass k at time t/N^2 should therefore converge to $(1+\varphi(t))^{-1}\nu_{p(t)}(k) = c_t(k)$.

Proof of Theorem 1.1: Since all quantities are bounded, we only have to establish convergence in L^2, say. It is convenient to interpret the coalescent $\mathcal{X}^{[N]}$ as the ranked cardinalities of a partition-valued process on the set $\{1, \ldots, N\}$. We label the N particles of unit mass from the starting configuration $\mathcal{X}^{[N]}(0)$ by $r_1, \ldots, r_N$. Then, at time t, $\mathcal{X}^{[N]}(t)$ consists out of $N - 2J^{[N]}(t)$ atoms which result from the coagulation of particles $\{r_i : i \in B_j^{[N]}(t)\}$, where $B_j^{[N]}(t)$, $j = 1, \ldots, N - 2J^{[N]}(t)$, is a (random) partition of $\{1, \ldots, N\}$ into non-empty subsets of odd cardinality. We denote by $C_i^{[N]}(t)$ the set

$B_j^{[N]}(t)$ that contains the index i. Now fix $t \geq 0$, and write $C_i = C_i^{[N]}(t/N^2)$. For every $k \in \mathbb{N}$,

$$\mathbb{E}\left[c_t^{[N]}(k)\right] = \frac{1}{Nk}\mathbb{E}\left[\sum_{i=1}^{N} \mathbb{1}_{\{|C_i|=k\}}\right] = \frac{1}{k}\mathbb{P}\left(|C_1| = k\right),$$

where in the last equality we have used the fact that the C_i are identically distributed. The probability on the right is given by the probability that the mass of an atom from $\mathcal{X}^{[N]}(t/N^2)$, picked uniformly at random among the non-zero components, is equal to k. Thus Lemma 1.3 shows that as $n \to \infty$, with $p(t)$ as above,

$$\mathbb{E}\left[c_t^{[N]}(k)\right] \to \frac{\nu_{p(t)}(k)}{k}.$$

Furthermore,

$$\begin{aligned}
\mathbb{E}\left[\left(c_t^{[N]}(k)\right)^2\right] &= \frac{1}{N^2k^2}\mathbb{E}\left[\sum_{i,j=1}^{N} \mathbb{1}_{\{|C_i|=k\}}\mathbb{1}_{\{|C_j|=k\}}\right] \\
&= \frac{N-1}{Nk^2}\mathbb{P}\left(|C_1| = k,\ |C_2| = k\right) + \frac{1}{Nk}\mathbb{P}\left(|C_1| = k\right).
\end{aligned}$$

Using again Lemma 1.3, we see that

$$\mathbb{E}\left[\left(c_t^{[N]}k\right)^2\right] \to \left(\frac{\nu_{p(t)}(k)}{k}\right)^2.$$

Altogether, this proves L^2-convergence. It remains to argue that the limit solves the equations (2). First note that since $\nu_{p(t)}$ is a probability distribution,

$$\sum_{k=1}^{\infty} c_t(k) = \frac{1}{1+\varphi(t)}. \tag{6}$$

Next, using that $\nu_{p(t)}$ has mean $(2p(t)-1)^{-1} = 1+\varphi(t)$,

$$\sum_{k=1}^{\infty} kc_t(k) = 1. \tag{7}$$

We therefore obtain for the part in (2) reflecting the disappearance of particles of mass k

$$-\frac{1}{2}c_t(k)\sum_{l,m=1}^{\infty} c_t(l)c_t(m)(k+l+m+3) = -\frac{1}{2}c_t(k)\left(\frac{2}{1+\varphi(t)} + \frac{k+3}{(1+\varphi(t))^2}\right).$$

The part which stands for the creation of particles of mass $k \geq 3$ is given by

$$\frac{1}{6}\sum_{l+m<k} c_t(l)c_t(m)c_t(k-l-m)(k+3) = \frac{(k+3)}{6(1+\varphi(t))^3}\mathbb{P}\left(H_{-3} = k\right),$$

where H_{-3} is the first hitting time of -3 of a random walk $\zeta = \{\zeta_m\}_{m\geq 0}$ on $\mathbb{Z}$ with $\zeta_0 = 0$ and i.i.d. increments $\mathbb{P}(\zeta_i - \zeta_{i-1} = 1) = 1 - p(t)$, $\mathbb{P}(\zeta_i - \zeta_{i-1} = -1) = p(t)$. By Kemperman's formula [8], for odd k,

$$\begin{aligned}\mathbb{P}(H_{-3} = k) = \frac{3}{k}\mathbb{P}(\zeta_k = -3) &= \frac{3}{k}\binom{k}{(k+3)/2} p(t)^{(k+3)/2}(1-p(t))^{(k-3)/2} \\ &= \frac{3(k-1)(1+\varphi(t))(2+\varphi(t))}{(k+3)\varphi(t)} c_t(k).\end{aligned}$$

Putting everything together, it remains to verify that for all positive integers k,

$$\frac{\mathrm{d}}{\mathrm{d}t} c_t(k) = \frac{1}{2}\left(\frac{(k-1)(2+\varphi)}{\varphi(1+\varphi)^2} - \frac{2}{(1+\varphi)} - \frac{(k+3)}{(1+\varphi)^2}\right) c_t(k).$$

This can directly be checked by differentiating $c_t(k)$, using that the derivative of φ satisfies

$$\varphi'(t) = \frac{2+\varphi(t)}{1+\varphi(t)}.$$

□

1.5 Some remarks

- As (7) shows, the mass is preserved over time, i.e. *gelation* does not occur. This phenomenon describes the emergence of particles formed by an infinite number of smaller particles, an event which is not incorporated into the equations (2). It can for example be observed when studying the multiplicative binary coagulation kernel $\tilde{\kappa}(k,l) = kl$.

- It is easy to see that there is a unique (differentiable) solution to (2) which preserves the mass. Namely, if $(\tilde{c}_t(k), k \in \mathbb{N}, t \geq 0)$ is any solution of (2), we note that due to the initial configuration, the total concentration of particles $\tilde{C}_t = \sum_{k=1}^{\infty} \tilde{c}_t(k)$ satisfies $\tilde{C}_0 = 1$. Further, it follows from the equations that $\tilde{C}_t$ is non-increasing in time and solves the ODE

$$\begin{cases} f'(t) = -f^2(t) - f^3(t), & t \geq 0, \\ f(0) = 1. \end{cases}$$

  In agreement with (6), the unique solution to this equation is given by $(1+\varphi(t))^{-1}$. Together with $\sum_{k=1}^{\infty} k\tilde{c}_t(k) = 1$, this determines recursively all $\tilde{c}_t(k)$.

- A common method to solve systems like (1) or (2) is to consider generating functions (for the concentrations). One obtains a nonlinear PDE which can sometimes be solved explicitly by the method of characteristics. Then one can try to use a Lagrange inversion formula to retrieve the concentrations. See e.g. Bertoin [5], where (binary) coagulation equations with limited aggregations are studied.

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